THE

MATHEMATICAL GAZETTE.

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SOUTHAMPTON AND DISTRICT BRANCH.

A MEETING of the Southampton Branch of the Mathematical Association took place on Saturday evening, November 12th, at King Edward VI. Grammar School. The President, Professor E. L. Watkin, M.A., of the Hartley College, was in the chair.

After the minutes had been read and two new Members elected, the President illustrated a method of teaching the Rule of Signs in Algebra to beginners by using squares of coloured cardboard, green on one side and white on the other, the green side representing positive quantities and the white side negative quantities.

By manipulating these squares Professor Watkin elucidated most clearly the operations of addition and subtraction in the case of both positive and negative quantities.

The attempt to elucidate multiplication and division was not so successful, for a most animated debate arose in which all present took part. Professor Watkin was overwhelmed with questions and suggestions, some relevant and others irrelevant to the subject in hand. The debate lengthened itself out to a late hour, and the conclusion was arrived at that all had something to learn about the Rule of Signs as applied to multiplication and division, and it was also unanimously agreed that this part of the subject bristled with difficulties too great for the consideration of beginners.

The Hon. Sec. suggested that the difficulty felt in elucidating multiplication and division arose from the fact that these operations have no objective existence as such, and therefore can only be experimentally dealt with as additions and subtractions. The Hon. Sec.'s views were not, however, acceptable to the meeting.

THE ARITHMETIC OF INFINITES.

A SCHOOL INTRODUCTION TO THE INTEGRAL CALCULUS.

BY T. PERCY NUNN, M.A., D.Sc.

DR. JOHN WALLIS, who lived through the greater part of the seventeenth century (1616-1703), was a typical child of that energetic age. Son of a Kentish parson, he began his academic career at Cambridge, but ended as Savilian professor at Oxford. He was a scholar who apparently read mathematics in Arabic as easily as he wrote on them in Latin; a

founder of the Royal Society; a grammarian; and (needless to say) a vigorous theologian. Into the middle of his famous *Algebra* he "spatchcocked" (with apologies for its irrelevance) one of the earliest modern inquiries in experimental psychology: in the course of a sleepless night he had amused himself by extracting mentally the square root of 3 to 20 places and, a month later, showed in the presence of a distinguished foreigner that he could remember the result correctly. Nor did these accomplishments exhaust his versatility. For as a supporter of the Parliamentarians he confounded the Royalists by his astonishing facility in unravelling their cipher messages. Those who are offended by his adherence to the cause of the Parliament may be reconciled by the knowledge that he was one of the remonstrants against the execution of the King—a defection which, happily, did not blind the authorities to his great merits when the Savilian professorship fell vacant in the following year.

Wallis's mathematical works fill almost the whole of three large Latin folios, the publication of which was completed in Oxford in 1699. Of these the *Arithmetica Infinitorum* (first published in 1655) is the best expression of his enterprising and vigorous mind. Though set out (like most works in that age) in the academic panoply of Propositions, Scholia and Corollaries, it is essentially practical and heuristic in spirit. Is is in fact an adventurous dash into attractive but little explored mathematical territory. For this reason it offers admirable examples of method to the teacher who has learnt that his subject should be presented, not as a static body of "truths," but as a progressive attempt to bring ordered thinking to bear upon questions of practical or intellectual interest.

In addition, Wallis's results are of the highest importance both intrinsically and for their significance in the history of mathematics. The *Arithmetica Infinitorum* had a marked influence upon the greater genius of Isaac Newton, and its discussions formed the starting point of many of his most important discoveries. Speaking generally, the problems which formed the motive of Wallis's inquiries were problems of integration, and his attempts to solve them form a most interesting and extraordinarily simple introduction to that important province of mathematics. From the point of view of the teaching (as distinguished from the actual advancement) of mathematics it is unfortunate that the more perfect work of Newton and his successors should have diverted attention from a treatment so excellently suited to the needs of the beginner. One of the main objects of this article is to draw to it the attention of the modern teacher who believes that some acquaintance with the methods and results of the calculus should be possessed by every educated person, but does not see how to impart it during the school age and under school conditions.

The work falls naturally into three sections. In the first Wallis investigates the problem which we should now-a-days describe as the "integration of x^n when n is a positive whole number"—his motive being the need of such results for determining the lengths of curves and the magnitudes of areas and volumes. In the next section his extension of the investigation to reciprocals, such as $\frac{1}{x^3}$, and to roots, such as $\sqrt[3]{x^2}$, leads him to the beautiful invention of negative and fractional indices. In the third section he uses his methods to reach a striking and entirely novel evaluation of π. In this article (which is based upon the folio edition of 1699), it is proposed to give a brief exposition of Wallis's argument under headings corresponding to these sections, and to complete it by an account of the way in which, from one of his discussions, Newton obtained the idea of the binomial theorem. To make the description at once compendious and intelligible to the ordinary reader, many elements must be excluded that to one who is familiar with seventeenth century mathe-

matics give the very "form and pressure" of the time. But except in one or two instances, to which attention is called, there has been no tampering with important details of the argument, and the spirit of the old author's method has been preserved throughout. It may be added that the changes referred to have been made because the present writer is convinced that Wallis's work, so far from having merely antiquarian interest, is of living value to the ordinary teacher of mathematics. It seems, therefore, justifiable to present his work with modifications that may make its usefulness still more apparent.

I. Fundamental Ideas.

The assumption upon which Wallis's evaluation of areas and volumes is based was adopted from Cavalieri's *Geometria Indivisibilibus Continuorum* (1635). A plane figure (such as a triangle) may be thought of as built up of an infinite number of parallel straight lines; a solid (such as a pyramid) of an infinite number of parallel plane figures—straight line and plane figure being conceived as having an excessively small but definite and indivisible thickness. In what follows this assumption is replaced by the notion, more accordant with modern ideas, that a plane figure may be regarded as the "limit" of a series of rectangles or parallelograms whose number is indefinitely increased and that a solid may similarly be thought of as the limit of a number of slabs of indefinitely small thickness.*

With this change Wallis's general argument may at once be made clear by a simple example. Let DB (fig. 1) be a rectangle whose base AB is divided into any number of equal parts. On these segments of the base let a number of rectangles be erected, a, b, c, ..., the height of the first being zero and the heights of the others in arithmetical progression. Let the height of the last rectangle of the series be BC, the height of the original figure. We are now to determine the ratio of the total area of the series of rectangles a, b, c, ... to DB. Calling the area of b unity, the areas of the other constituent rectangles are obviously members of the series of natural numbers. It is obvious, too, that the area of DB may be regarded as the sum of the series of equal rectangles whose number is that of the constituents a, b, c, ... and whose area is that of the last of these constituents. Supposing the number of parts into which AB is divided to be on successive occasions 2, 3, 4, 5, ..., we have for the ratio in question:

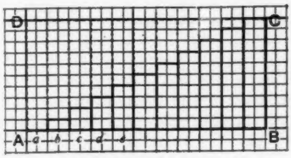

FIG. 1.

$$\frac{0+1}{1+1}=\frac{1}{2},$$

$$\frac{0+1+2}{2+2+2}=\frac{1}{2},$$

$$\frac{0+1+2+3}{3+3+3+3}=\frac{1}{2},$$

$$\frac{0+1+2+3+4}{4+4+4+4+4}=\frac{1}{2}, \quad \text{and so on.}$$

*It should, however, be remembered that Cavalieri and, after him, Wallis declared that the doctrine of "indivisibles" was intended to be merely a convenient form of expression of this same notion. (See Wallis, *Algebra*, Ch. 74, and Montucla, *Hist. des Mathématiques*, Pt. IV. Liv. I. Ch. VI.)

Thus we conclude that for any number of rectangles of which the area of the last is l,

$$\frac{0+1+2+3+\ldots+l}{l+l+l+l+\ldots+l}=\frac{1}{2}. \qquad \text{(A)}$$

It appears, then, that the ratio of the series of rectangles to the rectangle *DB* is always $\frac{1}{2}$, no matter how numerous the series may be. But when the series of rectangles is increased indefinitely, its limit is the triangle *ABC*. We conclude that the area of the triangle is one-half of that of the rectangle having the same altitude and base.

The arithmetical relationship just discussed can be applied with equal directness to the determination of many other areas and volumes. Wallis gives the volume of a paraboloid of revolution as a simple example of the power of his method. In the parabola *VPQ* (fig. 2), MP^2 bears a constant ratio to *VM*. But if the parabola rotated about its axis *VN*, *MP* would trace out a circle whose area would be proportional to MP^2 and therefore to *VM*. If an indefinite number of lines such as *MP* be imagined erected at equal distances along the axis, they will, on the revolution of the figure, trace out a series of circles whose areas form an arithmetical progression. Upon these circles a series of circular slabs can be supposed to rest whose limit is the paraboloid of revolution. Thus, by the arithmetical proposition before us, the total volume of the series of slabs will be one-half the total volume of a series composed of the same number of slabs each of which is equal to the last of the former series. That is, the volume of the paraboloid is one-half of the volume of the cylinder produced by the rotation of the rectangle *VRQ*.

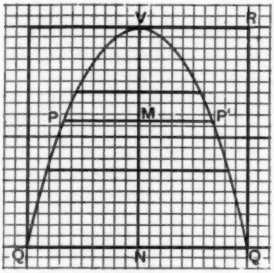

FIG. 2.

If we have (as in fig. 3) a series of rectangles whose heights (and areas) are successively proportional to the squares of the natural numbers, then we have as their limit the figure *ABC* of fig. 4, bounded by the parabola *AC*. The ratio of the total area of the series of increasing rectangles, to a series containing the same number of rectangles, but each equal to the last rectangle (*BC*) of the former series, can be found by calculation in the various cases as before :

$$\frac{0^2+1^2}{1^2+1^2}=\frac{1}{3}+\frac{1}{6},$$

$$\frac{0^2+1^2+2^2}{2^2+2^2+2^2}=\frac{1}{3}+\frac{1}{12},$$

$$\frac{0^2+1^2+2^2+3^2}{3^2+3^2+3^2+3^2}=\frac{1}{3}+\frac{1}{18},$$

$$\frac{0^2+1^2+2^2+3^2+4^2}{4^2+4^2+4^2+4^2+4^2}=\frac{1}{3}+\frac{1}{24}, \quad \text{etc., etc.}$$

The law governing the successive values of the ratio soon becomes evident. It is $\frac{1}{3}$ increased by a fraction which becomes proportionally smaller as the number of terms increases. We may, in fact, write

$$\frac{0^2+1^2+2^2+3^2+\ldots+l^2}{l^2+l^2+l^2+\ldots\ldots+l^2}=\frac{1}{3}+\frac{1}{6l}. \qquad \text{(B)}$$

It is clear that the larger the number of rectangles, the more closely the ratio approximates to $\frac{1}{3}$. We deduce, therefore, that the area of the figure ABC (fig. 4) is $\frac{1}{3}$ of the whole rectangle DB.

The volumes of the cone and pyramid can readily be determined by the aid of this same arithmetical result.

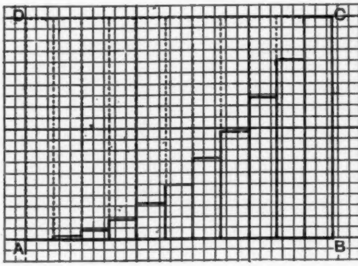

FIG. 3.

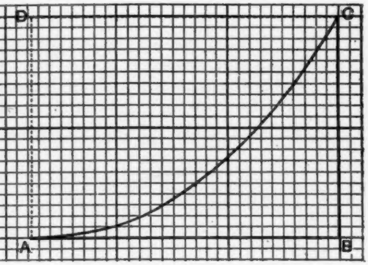

FIG. 4.

The case of a series of cubes of the natural numbers can be dealt with with equal ease. *E.g.*

$$\frac{0^3+1^3}{1^3+1^3}=\frac{1}{4}+\frac{1}{4},$$

$$\frac{0^3+1^3+2^3}{2^3+2^3+2^3}=\frac{1}{4}+\frac{1}{8},$$

$$\frac{0^3+1^3+2^3+3^3}{3^3+3^3+3^3+3^3}=\frac{1}{4}+\frac{1}{12},$$

$$\frac{0^3+1^3+2^3+3^3+4^3}{4^3+4^3+4^3+4^3+4^3}=\frac{1}{4}+\frac{1}{16}, \quad \text{etc.} = \text{etc.}$$

Here the rule appears to be that

$$\frac{0^3+1^3+2^3+\ \ldots\ldots\ +l^3}{l^3+l^3+l^3+l^3+\ldots+l^3}=\frac{1}{4}+\frac{1}{4l}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(c)}$$

so that the limiting value of the ratio is $\frac{1}{4}$.

It will be observed that in (B) and (C) Wallis makes no attempt to show that the expression he has reached will hold good for all values of l. He is content to accept it as an induction in the ordinary sense of the term—that is as a generalisation from a few particular cases. The teacher who is not satisfied to follow Wallis's procedure here, can, of course, easily prove that if either of the results in question holds good for any one value of l, it will hold good for the next, and so on universally. Thus, in the case of (B), suppose that for some value of l

$$\sum_0^l r^3/(l+1)l^3=\frac{1}{4}+\frac{1}{4l},$$

i.e.

$$\sum_0^l r^3=\frac{(l+1)^2l^2}{4};$$

then, if the rule holds good also for the next value of l, we should have

$$\sum_0^{l+1} r^3=\frac{(l+2)^2(l+1)^2}{4}.$$

But
$$\sum_0^{l+1} r^3 = \sum_0^{l} r^3 + (l+1)^3$$
$$= \frac{(l+1)^2 l^2}{4} + (l+1)^3$$
$$= \frac{(l+2)^2 (l+1)^2}{4}.$$

Hence, since the rule has been shown to hold good when $l=1, 2, 3, 4$, it holds good universally.

The results with series composed of higher powers of the natural numbers do not admit of such easy analysis as the foregoing. Thus, in the case of the fourth powers, the first few values of $\sum_0^l r^4/(l+1)l^4$ will be found to be $\frac{1}{5}+\frac{3}{10}$, $\frac{1}{5}+\frac{37}{240}$, $\frac{1}{5}+\frac{83}{810}$, $\frac{1}{5}+\frac{49}{640}$. The task of discovering a function of the number of terms of which these fractions are successive values would be unprofitably troublesome and tedious. The teacher will probably be contented with asking the class to verify that they are all represented by the expression
$$\frac{1}{5}+\frac{1}{10l}+\frac{1}{30l^2}-\frac{1}{30l^3}. \quad \text{.................................(D)}$$

Similarly, in the case of the series of fifth powers it must be regarded as sufficient to show that the first few values of $\sum_0^l r^5/(l+1)l^5$ are all given by the function
$$\frac{1}{6}+\frac{1}{3l}+\frac{1}{12l^2}-\frac{1}{12l^3}. \quad \text{.................................(E)}$$

With the results (A) to (E) before us, we shall find no difficulty in assuming with Wallis that if the number of terms in the numerator and denominator be supposed to increase indefinitely
$$\frac{0^n+1^n+2^n+\ldots+l^n}{l^n+l^n+l^n+\ldots+l^n}=\frac{1}{n+1}$$
for all the values 1, 2, 3, ... of n.

It is obvious that this simple rule can be applied not only to the evaluation of areas and volumes, but also to the solution of all problems in mechanics and physics, which are usually made to depend upon a knowledge of the "integration" of x^n. Thus, to find the position of the centre of gravity of a cone or pyramid, we may argue as follows: Let the solid be supposed divided by planes parallel to the base into a very large number of slabs of equal thickness. Then, since the mass of a slab is proportional to the square of its distance from the vertex O, its moment about O is proportional to the cube of the same distance. It follows that the total moment about the vertex is $\frac{1}{4}$ of the moment of an equal number of slabs all equal to the largest slab, and supposed all to be at the greatest distance from the vertex. That is, the total moment is the moment of a mass equal to that of the cylinder (or prism) which has the same base and height as the cone (or pyramid), and is placed at the point $\frac{1}{4}$ of the height from O. But the mass of the cone (or pyramid) is only $\frac{1}{3}$ of the mass of this cylinder (or prism). Hence the moment of the solid about O is that which its whole mass would have if it were placed $\frac{3}{4}h$ from O.

In the same way, we may determine the moment of inertia of a circular disc about its axis. The disc being supposed divided up into a large number (n) of concentric rings of equal depth, we know that the mass of a given ring is proportional to its radius. Its moment of inertia is proportional, therefore, to the cube of the radius. It follows that the total moment of inertia of the disc is $\frac{1}{4}$ of the moment of inertia of a cylinder composed of

$(n+1)$ rings equal to the largest of the actual rings. If the mass of this cylinder is m' and its radius (which is of course the radius of the disc) is r, then the moment of inertia is $\frac{1}{2}m'r^2$. But, since the masses of the rings which compose the disc are in A.P., their total mass is $\frac{1}{2}$ of the mass of a cylinder built up of rings equal to the largest of them. Thus, if the mass of the disc is m, $m'=2m$. Then we have

$$\text{moment of inertia of disc} = \tfrac{1}{2}mr^2.$$

There is an alternative way of expressing Wallis's results which is often useful, and is more in accord with modern usage. Let the ordinate of a curve be proportional to the n^{th} power of its distance x from the vertex. We may represent it, then, by the expression px^n, p being a constant. If the length of the base of the curve is x_1 the final ordinate is px_1^n, and the area of the rectangle with the final ordinate as side is $px_1^n \times x_1$, *i.e.* px_1^{n+1}. We have learnt from Wallis that the area included between the curve, its base and the final ordinate is to the rectangle as 1 to $n+1$. We have, therefore,

$$\text{area} = \frac{1}{n+1}px_1^{n+1}.$$

This result can be applied (among other purposes) to finding the area of a curve whose ordinate at a given point is expressed as a series. Thus if

$$y = a + bx + cx^2 + dx^3 + \ldots\ldots\ldots\ldots \quad \text{(i)}$$

and x_1 is the distance of the final ordinate from the vertex, we have

$$\text{area} = ax_1 + \tfrac{1}{2}bx_1^2 + \tfrac{1}{3}cx_1^3 + \tfrac{1}{4}dx_1^4 + . \ \ldots\ldots\ldots\ldots \text{(ii)}$$

For the final ordinate may be supposed divided into segments representing the various terms of series (i), and the rectangles of which these various segments are the terminal sides will have areas which are measured respectively by the corresponding terms of series (ii).*

II. Fractional Indices.

The first three cases of Wallis's rule would suffice for the solution of all ordinary problems in mensuration or physics. It is necessary, however, to be convinced of its generality in order to follow the argument by which he extends its application from series of powers to series of roots of the natural numbers. In its dependence upon (ordinary) induction and analogy, this argument is very characteristic of Wallis's favourite methods of procedure. It runs as follows:

We have seen that, as the number of terms increases, the ratio of the sum of the series $0^n + 1^n + 2^n + \ldots + l^n$ to the sum of an equally numerous series of terms, each equal to the greatest term l, constantly approaches $\frac{1}{n+1}$. That is, if we consider cases in which the indices of the powers (*i.e.* the successive values of n) form the arithmetical progression 1, 2, 3, ..., the denominators of the limiting ratios form the arithmetical progression 2, 3, 4, ... From this we may first draw the conclusion that if the limiting ratio were 1 : 1, the series must be represented as $0^0 + 1^0 + 2^0 + \ldots + l^0$. But if for any large value of l we have

$$0^0 + 1^0 + 2^0 + 3^0 + \ldots + l^0 = l^0 + l^0 + l^0 + l^0 + \ldots + l^0$$

(the number of terms in each expression being the same), it is clear that

* Applied to the squaring of the hyperbola $y = 1/(1+x)$, this method led Mercator (*Logarithmotechnia*, 1668) to the invention of "hyperbolical" logarithms—that is, logarithms to base e. $\text{Log}_e(1+x)$ is the area between the ordinates whose abscissae are 1 and x.

$0^0=1^0=2^0=3^0=\ldots=l^0$. But unity remains unchanged when raised to any power. We must suppose, therefore, that each term in the series

$$0^0+1^0+2^0+3^0+\ldots+l^0$$

is equal to 1.*

Thus we may now say that if the power of the series is a term of the A.P. 0, 1, 2, 3, 4, ..., the consequent of the ratio is the corresponding term in the A.P. 1, 2, 3, 4, 5, ...

Let us next consider the series $0+1+8+27+\ldots$. This consists of the cubes of the natural numbers, and the ratio characteristic of it is the ratio $1:4$. We can suppose the series to have been reached by three stages represented by the series

$$0+1+2+\ \ 3+\ \ 4+\ldots,$$
$$0+1+4+\ \ 9+16+\ldots,$$
$$0+1+8+27+64+\ldots,$$

and the characteristic ratios to have been reached by corresponding stages,

$$1:2,\ \ 1:3,\ \ 1:4.$$

But if we take the given series as the starting point instead of the goal of our manipulations, then the terms of the series $0+1+2+3+4+\ldots$ must be thought of as the cube *roots* of the terms of $0+1+8+27+64+\ldots$ and the terms of $0+1+4+9+16+\ldots$ as the *squares* of these cube roots. Meanwhile the consequents (2 and 3) of the characteristic ratios of these derived series may be thought of as obtained by interpolating two arithmetic means between 1 and 4.

Similarly, if we start with the series $0+1+16+81+256+\ldots$ whose characteristic ratio is $1:5$, the terms of the earlier series may be thought of as respectively the fourth roots, the squares of the fourth roots, and the cubes of the fourth roots of the terms of the given series, while the consequents of their characteristic ratios (2, 3, 4) may be found by interpolating three arithmetic means between 1 and 5.

Let us next start with the series of squares of the natural numbers and obtain new series by taking the squares, cubes, etc., of its terms. Then it is clear that we shall obtain series whose characteristic ratios are $1:3$, $1:5$, $1:7$, etc. Conversely, if we start with the series $0+1+64+729+\ldots$ (*i.e.* with the series of sixth powers whose characteristic ratio is $1:7$), then the series of second powers and the series of fourth powers are obtained respectively by taking the cube roots and the square roots of the terms of given series. And, just as in the simpler example, the consequents of the ratios characteristic of these series are the two arithmetic means between 1 and 7 (the consequent of the characteristic ratio of the given series).

It seems evident from these examples that we have a general rule that may be formulated as follows. Start with a series whose characteristic ratio is $1:R$, and obtain from it $(n-1)$ new series by first taking the n^{th} roots of each of its terms and then raising the numbers so obtained to the $2^{\text{nd}}, 3^{\text{rd}}, \ldots (n-1)^{\text{th}}$ powers successively. The consequents of the characteristic ratios of these derived series will be the $(n-1)$ arithmetic means between 1 and R.

Now, argues Wallis, there appears no reason why we should not apply this rule to the series of natural numbers itself. For example, let us obtain the series $\sqrt{0}+\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots$ by taking the square roots of the terms of $0+1+2+3+\ldots$. Then, by the rule, since the characteristic ratio of the initial series is $1:2$, the consequent of that of the derived series must be the arithmetic mean between 1 and 2—*i.e.* $1\frac{1}{2}$. Moreover, since the index

* This proof is not explicitly given by Wallis, but it is implied at several points in his argument.

of the series and the consequent of the characteristic ratio are corresponding terms of the arithmetical progressions

$$0,\ 1,\ 2,\ 3,\ 4, \ldots$$

and

$$1,\ 2,\ 3,\ 4,\ 5, \ldots,$$

it is evident that if we interpolate in the second progression a consequent $1\frac{1}{2}$ between 1 and 2, there must be interpolated a corresponding index between 0 and 1 in the first progression. Thus we find that the square root of a number may be represented as a power with index $\frac{1}{2}$.

Similarly, if we take the cube roots of the natural numbers, and afterwards square the results, we shall obtain the two series

$$\sqrt[3]{0}+\sqrt[3]{1}+\sqrt[3]{2}+\sqrt[3]{3}+\sqrt[3]{4}+\ldots$$

and

$$\sqrt[3]{0^2}+\sqrt[3]{1^2}+\sqrt[3]{2^2}+\sqrt[3]{3^2}+\sqrt[3]{4^2}+\ldots.*$$

By the rule, the consequents of the characteristic ratios of these derived series must be the two arithmetic means between 1 and 2—*i.e.* $1\frac{1}{3}$ and $1\frac{2}{3}$. Moreover, the corresponding indices must be the two arithmetic means between 0 and 1—*i.e.* $\frac{1}{3}$ and $\frac{2}{3}$.

The generalisation of these results is obvious.

(1) A number of the form $\sqrt[a]{r}$ (where a is of course a positive whole number) can be expressed as a power in the form $r^{\frac{1}{a}}$; while a number of the form $\sqrt[a]{r^b}$ or $(\sqrt[a]{r})^b$ can be expressed in the form $r^{\frac{b}{a}}$.

(2) The characteristic ratio of the series $0^n+1^n+2^n+3^n+\ldots$ is $1/(n+1)$, whether n is an integer, a fraction of the form $\frac{1}{a}$ or a fraction of the form $\frac{b}{a}$.

The validity of this extension of a rule actually established only for certain integral powers of the natural numbers, is in certain cases susceptible of verification. The semi-parabola VPN is represented in two positions in figs. 5 and 6. In fig. 5 the ordinates (such as pm) are proportional to the

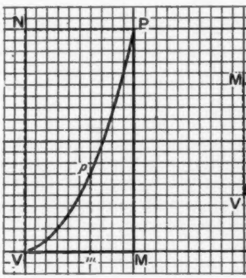

FIG. 5.

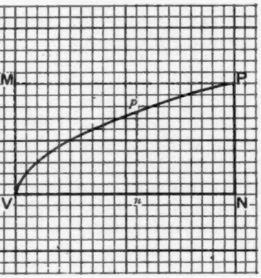

FIG. 6.

squares of the abscissae. It follows by the original rule that the complementary area VMP is $\frac{1}{3}$ of the parallelogram MN. But in fig. 6 the ordinates (such as pn) are proportional to the square *roots* of the abscissae. In accordance with the extension of the rule the area VPN should be $\frac{1}{1\frac{1}{2}}$, that is, $\frac{2}{3}$ of the parallelogram MN. This result is, obviously, concordant with the former.

* For $\sqrt[3]{r^2}=(\sqrt[3]{r})^2$, and is more conveniently written and printed.

III. Negative Indices.

The principle of continuity (or of interpolation as Wallis calls it), which has brought us to admit fractional indices, leads us with still more ease to the notion of a negative index. Take, for example, the two series

$$0^p+1^p+2^p+3^p+4^p+\dots\dots \quad \text{(i)}$$

and

$$0^q+1^q+2^q+3^q+4^q+\dots\dots \quad \text{(ii)}$$

in which p and q are integral (and of course positive) indices, and form from them a new series by dividing each term of (i) by the corresponding term of (ii). Three cases arise. The first is when p is greater than q. In this case the derived series is

$$0^{p-q}+1^{p-q}+2^{p-q}+3^{p-q}+4^{p-q}+\dots.$$

The second case is when p and q are equal. The division will now yield the series

$$1+1+1+1+1+\dots.*$$

In the third case p is less than q, and the derived series will be a series of reciprocals

$$\frac{1}{0^{q-p}}+\frac{1}{1^{q-p}}+\frac{1}{2^{q-p}}+\frac{1}{3^{q-p}}+\dots.$$

Now suppose that $p-q$ is, to begin with, 4, and that while p remains constant in value q begins to increase in value by unit steps. Then, the indices of the derived series will be successively the terms of the decreasing progression 4, 3, 2, 1, and the consequents of the characteristic ratios the corresponding terms of the progression 5, 4, 3, 2. Then, in accordance with the principle of continuity, we must suppose the second case and the successive results in the third case to be represented by corresponding terms which continue the two progressions in the sense in which they have been begun, *i.e.*

$$\begin{array}{l} 4,\ 3,\ 2,\ 1,\ 0,\ -1,\ -2,\ -3,\dots \\ 5,\ 4,\ 3,\ 2,\ 1,\ \ \ 1,\ -1,\ -2,\dots. \end{array}$$

Thus the series $1+1+1+1+\dots$ may (as we have already seen) be supposed to be equivalent to $0^0+1^0+2^0+\dots$; while such a series as $\frac{1}{0^3}+\frac{1}{1^3}+\frac{1}{2^3}+\dots$ must be capable of representation in the form $0^{-3}+1^{-3}+2^{-3}+\dots$, and must have the characteristic ratio $1:-2$, that is $\frac{1}{1-3}$.

Finally, if we were justified in interpolating positive fractional indices between the positive integral indices, and in making corresponding interpolations in the series of consequents of the ratios, we must be equally justified in completing the series of indices and consequents by the interpolation of corresponding negative fractional terms. For example, the series of reciprocals

$$\frac{1}{\sqrt[3]{0}}+\frac{1}{\sqrt[3]{1}}+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+\dots$$

can, no doubt, be validly represented in the form

$$0^{-\frac{1}{3}}+1^{-\frac{1}{3}}+2^{-\frac{1}{3}}+3^{-\frac{1}{3}}+\dots,$$

and its characteristic ratio must be $\frac{1}{1-\frac{1}{3}}$ or $\frac{3}{2}$.

Thus our original rule, that if the index of the series is n the characteristic

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* Analogy with the results of the other divisions justifies us here in assuming that

$$\frac{0^p}{0^p}=1.$$

ratio is $\frac{1}{n+1}$ has now been so extended that we can give a meaning to it, and find a use for it for all rational values of n integral or fractional, positive or negative.

There is, however, an outstanding problem which Wallis with all his sagacity did not solve.* What is the meaning of the negative ratio that follows from such an index as -3? Wallis got as far as follows. Let the dotted curve $A'PB'$ in fig. 7 be a rectangular hyperbola. That is, let the ordinates (such as $p'm$) be inversely proportional to the abscissae (such as Om). Any point P being taken on the curve, what is the measure of the area $A'PMOY$? As in former cases, we must suppose the base OM to be divided into an indefinitely large number of equal parts by points at which ordinates are to be erected, and we must determine the limit of the ratio of the sum of these ordinates to the sum of an equal number each equal to the last one, PM. The result will tell us the ratio of the hyperbolic area to the rectangle PO. But the heights of the ordinates at distances 0, 1, 2, 3, ... from O are given by the terms $0^{-1}, 1^{-1}, 2^{-1}, 3^{-1}, \ldots$. Then, in accordance with our rule, the ratio required is $\frac{1}{1-1}$, *i.e.* $\frac{1}{0}$ or ∞. Thus the hyperbolic area $A'PMOY$ is infinitely greater than the rectangle PO.

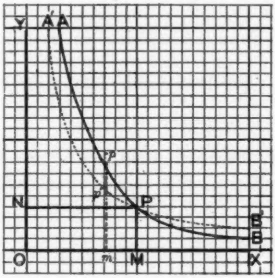


FIG. 7.

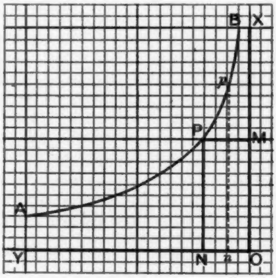


FIG. 8.

Next consider a curve APB, passing through the same point P, in which the ordinates (such as pm) are inversely proportional to the *cubes* of the abscissae (such as Om). That is, let the equation of the curve be $x^3y=c$. Our determination of the area $APMOY$ must now depend upon the properties of the series $0^{-3}+1^{-3}+2^{-3}+3^{-3}+\ldots$. But the characteristic ratio of this series is $\frac{1}{-3+1}$, *i.e.* $\frac{1}{-2}$. Our rule tells us, then, that the area in question is $\frac{1}{-2}$ of the rectangle OP.

In his attempt to interpret this result, Wallis could say only, that since -2 is less than 0, and the ratio $\frac{1}{0}$ is infinity, the ratio $\frac{1}{-2}$ must be greater still than infinity (*ratio plusquam infinita*). It was left for the French mathematician Varignon (1654-1722) to give a more satisfactory account of the matter. Varignon pointed out that if we drop the minus sign, the numerical value of the ratio is a measure of the area of the space $BPMX$ to the right of the terminal ordinate PM. This is best seen by turning fig. 7 through a right angle into the position shown in fig. 8, so that the ordinates

* See Montucla, *Histoire des Mathématiques*, Pt. IV. Liv. VI. Ch. I.

of the curve in its former position become abscissae, and conversely. Since in the original position the ordinates were inversely proportional to the cubes of the abscissae, the ordinates in the new position (such as pn) must be inversely proportional to the cube *roots* of the abscissae (such as On). The area $XONBP$ depends therefore on the properties of the series

$$0^{-\frac{1}{3}}+1^{-\frac{1}{3}}+1^{-\frac{1}{3}}+2^{-\frac{1}{3}}+\ldots,$$

whose characteristic ratio is $\dfrac{1}{1-\frac{1}{3}}=\dfrac{3}{2}$. That is, the area of $XONPB$ is one and a half times the rectangle PO. Deducting the area of PO, we have that the area of $XMPB$ is one-half of PO. In this case, then, the area calculated for the space $APMOY$ to the left of the ordinate PM (fig. 7) is, with a change of sign, the area of the space $BPMX$ to the right of that ordinate.

It is easy to show that this result will always hold good. Let the equation of the curve in its original position be $y=\dfrac{c}{x^m}$, where m is an integer or a fraction greater than unity. Then the ratio of the area $APMOY$ to the rectangle PO is $\dfrac{1}{-m+1}$, a negative result which may be written $-\dfrac{1}{m-1}$. Now turn the curve into the position in which it is most conveniently described as $x=\dfrac{c}{y^{\frac{1}{m}}}$. We then obtain for the ratio of the area $BPNOX$ to the rectangle PO the value $\dfrac{1}{1-\dfrac{1}{m}}$, *i.e.* $\dfrac{m}{m-1}$, a positive result. Hence the ratio of $BPMX$ to PO is $\dfrac{m}{m-1}-1=\dfrac{1}{m-1}$, the former ratio with the sign reversed. We conclude that the appearance in these calculations of a negative ratio must be taken as the indication that we are measuring a *negative* area—that is, the area on the other side of the ordinate.

(*To be continued.*)

THE TEACHING OF SPECIFIC VOLUME AND DENSITY.

By F. G. Daniell.

(*Abstract of Paper read before the London Branch of the Mathematical Association on 22nd October*, 1910.)

The purpose of the author was to urge that specific volume should precede density, because (1) the idea of specific volume is more important and fruitful than the idea of density; (2) the notion of specific volume is easier to acquire. Dealing first with the second reason, evidence was quoted from experienced science teachers and examiners to the effect that boys frequently failed to grasp the *idea* of density although they could formulate a correct definition, and carry out and describe a density determination with unimpeachable correctness. On the other hand the author had found that the idea of density had been readily acquired where the class had approached the subject through a preliminary lesson on specific volume. A short demonstration was then given of a method by which the relative specific volumes of substances or their comparative "roominess" (*Raümigkeit*), *i.e.* the volume occupied by 1 lb. of each, could be readily and effectively brought home to the class. (In subsequent discussion the chairman suggested the word 'roomage' in place of specific volume.)

Continuing the demonstration, the author illustrated the value of the specific volume idea. Thus the fundamental ideas in the theory of balancing columns of liquids and of gases were readily and simply educed, with important corollaries of value in studying convection and the relation of specific volume to temperature. The expansion of isotropic solids and liquids was expressed as an increase of specific volume with rise of temperature. For any "perfect" gas the specific volume varies as the absolute temperature. The author said that the idea of density might be introduced whenever it was wanted; but thought that there were advantages in its introduction in connection with Boyle's law. He preferred Boyle's statement that the spring, or elasticity, or pressure, of a gas varies as the density, and would plot observations thereon as a straight-line graph. It would be quite easy for those who preferred to use the hyperbolic relation to state the law thus —Specific Volume × Pressure = Constant. Flotation problems could be treated very simply, *e.g.* the specific volume of cork is four times that of water, hence the volume of a floating cork is four times the volume of the displaced water.

Investigators at the growing points of physics and chemistry usually coordinated changes of temperature, magnetisation, and molecular state with specific volume rather than with density. A simple diagram illustrating the relation of specific volume of water to temperature was exhibited. [Mathematical teachers were warned of the disasters which befall examiners in chemistry who assume that the volume of a mixture of liquid can be calculated from a knowledge of the separate volumes of the ingredients. This erroneous idea is common among boys of a mathematical bent.] An early acquaintance with specific volume made the more advanced studies easier to the higher classes in schools, *e.g.* Lothar Meyer's curve of atomic volumes could be understood without delay. Ostwald's great authority lent support to the argument that specific volume was more important than density.

During the discussion, in which agreement with the paper was expressed by the speakers, the author emphasised the advantage in class-instruction which was gained by direct appeal to the eye—easy and direct in the case of specific volume, but only indirectly possible in the case of density.

MATHEMATICAL NOTES.

333. [K. 2. d.] (1) *The radical axis of the circumcircle and the Brocard circle.* (2) *The relative positions of these circles.*

(1) Let the harmonic conjugates of the symmedians AK, BK, CK with respect to the corresponding pairs of sides meet the third sides in L_1, L_2, L_3. Then the line $L_1L_2L_3$ is the radical axis of the circumcircle and the Brocard circle.

The trilinear equation of the Brocard circle is

$$ayz+bzx+cxy=R\tan\omega\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(ax+by+cz),$$

where ω is the Brocard angle. Hence the radical axis of this circle and the circumcircle $ayz+bzx+cxy=0$ is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$. Now where $x=0$, $y:z=-b:c$. Hence this line meets BC in L_1. Similarly it meets CA and AB in L_2 and L_3. Thus $L_1L_2L_3$ is the radical axis.

(2) The Brocard circle intersects the circumcircle in the circular points and the points in which the circles are cut by their radical axis. But these points are the points of intersection of $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$, and $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$, viz.

$a : b\omega : c\omega^2$ and $a : b\omega^2 : c\omega$, where ω is one of the imaginary cube roots of unity. Hence the circumcircle and the Brocard circle never intersect in real points. Therefore, since the Brocard circle passes through the centre of the circumcircle, we have the theorem :—the Brocard circle always lies within the circumcircle.

H. L. TRACHTENBERG.

334. [K. 2. c.] *Feuerbach's Theorem.*

Let ABC be a triangle; I, I_1, J the centres of the inscribed circle, the escribed circle touching BC, and the nine-point circle; L the middle point of BC; D, D_1 the points of contact of the inscribed and escribed circles with BC, and S the point in which AII_1 cuts BC*; U the middle point of the arc LP of the nine-point circle, where AP is the perpendicular from A on BC: let JU cut BC in X, and the nine-point circle again in U'; draw UD, UD_1 cutting the nine-point circle in K and K_1; join $U'K$, $U'K_1$.

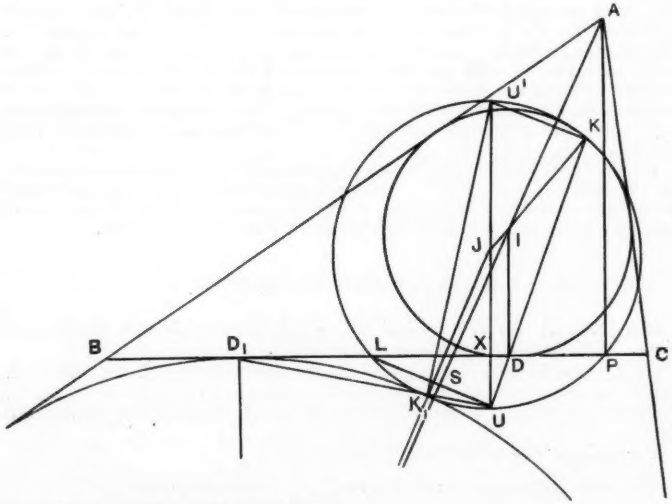
I. Since $\qquad I_1S : SI = r_1 : r = I_1A : IA;$

$$\therefore\ D_1S : SD = D_1P : PD.$$

Hence, since L is the middle point of DD_1;

$\therefore\ D_1L$ (or DL) is a mean proportional to LS and LP;

whence $\qquad SD . LP = LD . DP$ and $SD_1 . LP = LD_1 . D_1P.$



II. Since $\triangle$s ISD, LUX are similar;

$$\therefore\ ID : SD = LX : UX;$$

$$\therefore\ 2ID . UX = 2SD . LX = SD . LP = LD . DP; \text{ by I.}$$

$$\therefore\ 2ID . UX = UD . DK.$$

But $\qquad 2UJ . UX = UD . UK; \quad \because\ X, D, K, U$ are cyclic;

$$\therefore\ ID : UJ = DK : UK;$$

$\therefore\ \triangle$s IDK, JUK are similar.

* In the figure S should be on BC.

Hence, J, I, K are collinear, and $IK=ID$; and therefore the circles touch at K.

III. Similarly, $\because$ $\triangle$s I_1SD_1, LUX are equiangular;

$$\therefore\ 2I_1D_1 . UX = DK_1 . DU,$$
$$\text{and } 2UJ . UX = UK_1 . DU;$$
$$\therefore\ I_1D_1 : UJ = DK_1 : UK_1.$$

Hence K_1 is a centre of similitude, and the circles touch at K_1.

J. M. CHILD.

335. [M¹. h.] I. *To find a geometrical meaning for the triad of points* $P\equiv(x', y', z')$; $Q\equiv(x', \omega y', \omega^2 z')$; $R\equiv(x', \omega^2 y', \omega z')$.

Let ABC be the triangle of reference. Then it is a well-known fact and easily shewn that the circum-conic $\Sigma x'yz=0$ (1) is such that the lines joining A, B, C to the poles of BC, CA, AB respectively meet in the point $P\equiv(x', y', z')$. Now the polar of P with respect to this conic is $\Sigma\dfrac{x}{x'}=0$ (2). Solving for $x:y:z$ between the conic (1) and the line (2), we get as the points of intersection $Q\equiv(x', \omega y', \omega^2 z')$ and $R\equiv(x', \omega^2 y', \omega z')$. Hence the circum-conic through Q and R touches PQ and PR at Q and R respectively, and similarly for the others. It is easily seen that the Invariants Θ and Θ' vanish for this system of three conics. The above line and conic are generally called the Polar Line and Conic of P with respect to the triangle ABC. We shall call PQR an ω-triad with respect to the triangle ABC.

II. *If ABC be a triangle such that the polar conic of A with respect to a given Cubic Curve harmonically separates B and C, then the polar conics of B and C will harmonically separate CA and AB respectively.*

Refer the Cubic to ABC as triangle of reference, and let its equation be,

$$ax^3+by^3+cz^3+3a_1x^2y+3a_2x^2z+3b_0y^2x+3b_2y^2z+3c_0z^2x+3c_1z^2y+6mxyz=0.$$

Then the polar conic of $A\equiv(0, 0, 1)$ is

$$ax^2+b_0y^2+c_0z^2+2myz+2a_2zx+2a_1xy=0,$$

which meets BC in points given by $b_0y^2+2myz+c_0z^2=0$, and therefore harmonically separates BC if $m=0$, whence the result follows at once. We shall call ABC an Apolar Triangle with respect to the given Cubic Curve.

III. *If ABC be an Apolar Triangle inscribed in the Cubic Curve, and if PQR be an ω-triad of points with respect to ABC, then PQR will be an apolar triangle with respect to the Cubic.*

Since the Cubic circumscribes ABC, $a=b=c=0$, and since ABC is an Apolar Triangle, $m=0$. We thus get

$$a_1x^2y+a_2x^2z+b_0y^2x+b_2y^2z+c_0z^2x+c_1z^2y=0.$$

The polar conic of $P\equiv(x', y', z')$ will be

$$a_1(x^2y'+2xyx')+a_2(x^2z'+2zxx')+b_0(y^2x'+2xyy')+b_2(y^2z'+2yzy')$$
$$+c_0(z^2x'+2zxx')+c_1(z^2y'+2yzz')=0,$$

to which $Q\equiv(x', \omega y', \omega^2 z')$ and $R\equiv(x', \omega^2 y', \omega z')$ can easily be seen to be conjugate points.

IV. *If ABC be an Apolar Triangle inscribed in a Cubic, and if the tangents at A, B, C be concurrent, to find a geometrical interpretation for the coefficients of the Cubic referred to ABC as triangle of reference.*

Let us use that system of homogeneous coordinates in which the point of concurrency O of the tangents at A, B, C is (1, 1, 1).

Then the equation to the Cubic is of the form

$$\Sigma ax^2(y-z)=0. \qquad\qquad (1)$$

Plainly the point $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ lies on the Curve. The tangent thereat is without difficulty found to be

$$\Sigma x\left(\frac{1}{b}-\frac{1}{c}\right)=0, \qquad \text{.....................................(2)}$$

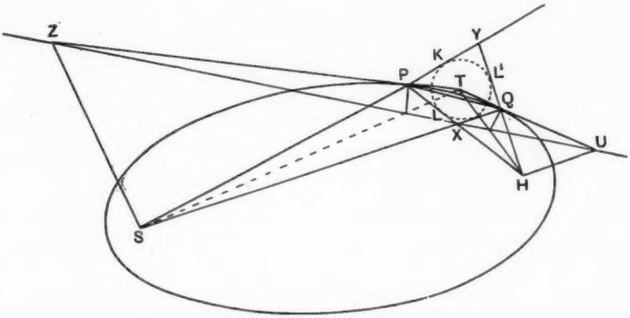
which passes through O. Now O lies on the Cubic by (1). Hence $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ is the point of contact of the fourth tangent from O to the Cubic.

W. P. MILNE.

336. [L^1. 4. 7.] *Tangents and foci of conics. Alternative proof of standard theorems.*

Let P, Q be two points on an ellipse of which the foci are S and H.

Since $SP+PH=SQ+QH$, a circle can be drawn to touch the quadrilateral formed by the focal distances. Let T be the centre of this circle.



Let SQ, HP cut at X, and let Z, U be the excentres opposite X of SPX, HQX.

The direction of the chord PQ is between those of ZT, TU. Therefore, in the limit when P, Q, T, X coincide, *the tangent at P is ZU, the exterior bisector of SPH.*

Again, in the figure PT, TQ are now seen to be the tangents at P and Q, and it follows that *the tangents from T subtend equal angles at either focus.*

Finally, because T is the excentre of SPX,

$$\angle PTS=\tfrac{1}{2}\angle PXS=\tfrac{1}{2}\angle QXH=\angle QTH,$$

and the tangents from T are equally inclined to the focal distances of T.

F. J. W. WHIPPLE.

337. [K. 2. e.] *The bisectors of the angles of a triangle ABC meet the opposite sides in X, Y, Z.*

The circle XYZ passes through the point of contact of the in- and nine-point circles.

If $L=0$, $M=0$, $N=0$ are the equations to the sides of any triangle inscribed in the triangle of reference, then the equation of any conic circumscribing the triangle LMN may be written $p\alpha L+q\beta M+r\gamma N=0$. For this equation is plainly satisfied by $\{M=0, N=0\}$, etc., since $\alpha=0$ when $M=0$ and $N=0$.

The equation to the circle XYZ may therefore be written

$$p\alpha(\alpha-\beta-\gamma)+q\beta(\beta-\gamma-\alpha)+r\gamma(\gamma-\alpha-\beta)=0.$$

Applying the conditions that this should represent a circle:

$$\begin{aligned} q\sin^2C+r\sin^2B+(q+r)\sin B\sin C \\ =r\sin^2A+p\sin^2C+(r+p)\sin C\sin A \\ =p\sin^2B+q\sin^2A+(p+q)\sin A\sin B. \end{aligned}$$

From which we deduce

$$p:q:r=a+2(a+b+c)\cos A:b+2(a+b+c)\cos B:c+2(a+b+c)\cos C.$$

The equation to the circle XYZ is therefore

$$\begin{aligned} (a+4s\cos A)\alpha(\alpha-\beta-\gamma)+(b+4s\cos B)\beta(\beta-\gamma-\alpha) \\ +(c+4s\cos C)\gamma(\gamma-\alpha-\beta)=0. \end{aligned}$$

The above theorem now follows from the identity,

$$\begin{aligned} \sin^2\tfrac{1}{2}(B-C)-\sin^2\tfrac{1}{2}(C-A)-\sin^2\tfrac{1}{2}(A-B) \\ =2\cos\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}(C-A)\sin\tfrac{1}{2}(A-B). \end{aligned}$$

For the coordinates of the point of contact of the inscribed and nine-point circles are given by

$$\alpha':\beta':\gamma'=\sin^2\tfrac{1}{2}(B-C):\sin^2\tfrac{1}{2}(C-A):\sin^2\tfrac{1}{2}(A-B).$$

Hence $\alpha'(\alpha'-\beta'-\gamma')=\sin(B-C)\sin\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}(C-A)\sin\tfrac{1}{2}(A-B)$
$=\kappa\sin(B-C)$, say.

Similarly $\beta'(\beta'-\gamma'-\alpha')=\kappa\sin(C-A)$, $\quad\gamma'(\gamma'-\alpha'-\beta')=\kappa\sin(A-B)$.

Therefore $\Sigma(a+4s\cos A)\alpha'(\alpha'-\beta'-\gamma')$
$=\kappa\Sigma a\sin(B-C)+4\kappa s\Sigma\cos A\sin(B-C)$,

and both terms in the latter expression vanish. JOHN H. LAWLOR.

338. [K. 6. a.] *On the Centre of a Circle in Trilinears.*

The equation to the circle in trilinears may be written

$$\begin{aligned} \lambda_1^2(x_2^2+x_3^2+2x_2x_3\cos A_1)+\lambda_2^2(x_3^2+x_1^2+2x_3x_1\cos A_2) \\ +\lambda_3^2(x_1^2+x_2^2+2x_1x_2\cos A_3)+\lambda^2(a_1x_1+a_2x_2+a_3x_3)^2=0. \end{aligned}$$

For (1) it is homogeneous of the second degree;

(2) it contains three independent constant ratios;

(3) it is satisfied by the circular points at infinity.

This form shows that the circle is the locus of a point whose pedal triangle has three sides p_1, p_2, p_3 obeying the relation

$$\lambda_1^2p_1^2+\lambda_2^2p_2^2+\lambda_3^2p_3^2+4\lambda^2\Delta^2=0,$$

i.e. a point whose distances from the vertices obey the relation

$$\delta_1^2\sin^2A_1\lambda_1^2+\delta_2^2\sin^2A_2\lambda_2^2+\delta_3^2\sin^2A_3\lambda_3^2+4\Delta^2\lambda^2=0.$$

Hence the centre of the circle is the mean centre of the vertices for multiples

$$\lambda_1^2\sin^2A_1:\lambda_2^2\sin^2A_2:\lambda_3^2\sin^2A_3.$$

If we identify this form with the more usual one,

$$*\,2[l(x_2x_3a_1+x_3x_1a_2+x_1x_2a_3) + (l_1x_1+l_2x_2+l_3x_3)(a_1x_1+a_2x_2+a_3x_3)]=0,$$

we shall find, by eliminating λ^2, that

$$\lambda_1^2\sin^2 A_1 = a_1(l_2\cos A_3 + l_3\cos A_2 - l_1 - l\cos A_1),$$

and two similar equations.

Hence at once we have as equations determining the centre of a circle whose equation is expressed in the usual form,

$$\theta a_1 = l\cos A_1 + l_1 - l_2\cos A_3 - l_3\cos A_2,$$
$$\theta a_2 = l\cos A_2 + l_2 - l_3\cos A_1 - l_1\cos A_3,$$
$$\theta a_3 = l\cos A_3 + l_3 - l_1\cos A_2 - l_2\cos A_1,$$

where for actual values it is easily seen that $\theta = \dfrac{l}{R}$.

From this $\theta(l_1a_1+l_2a_2+l_3a_3) = l(l_1\cos A_1 + l_2\cos A_2 + l_3\cos A_3) + \Pi^2$,

where $\Pi^2 = l_1^2 + l_2^2 + l_3^2 - 2l_2l_3\cos A_1 - 2l_3l_1\cos A_2 - 2l_1l_2\cos A_3$.

Now, if p, ϖ be the distances of the given and circum-circles from their radical axis,

$$l_1x_1+l_2x_2+l_3x_3=0,$$

this formula gives $\theta p = \dfrac{l}{R}\varpi + \Pi$ or $\dfrac{l}{R}(p-\varpi) = \Pi$.

But $p - \varpi = \delta$, the distance between the centres.

Therefore $$\delta^2 = \frac{\Pi^2 R^2}{l^2}.$$

If m be the modulus of the circumcircle, lm is evidently the modulus of the given one; and therefore when we substitute the coordinates of the circumcentre in the equation of the given one whose radius is ρ, say,

$$lm(-R^2) + 2\Delta R(l_1\cos A_1 + l_2\cos A_2 + l_3\cos A_3) = lm(\delta^2 - \rho^2);$$

$$\therefore\ \rho^2 = R^2\left(\frac{\Pi^2}{l^2}+1\right) - \frac{2\Delta R}{lm}(l_1\cos A_1 + l_2\cos A_2 + l_3\cos A_3)$$

$$= R^2\left\{\frac{\Pi^2}{l^2} + 2\frac{l_1\cos A_1 + l_2\cos A_2 + l_3\cos A_3}{l} + 1\right\},$$

since $$m = -2R\sin A_1\sin A_2\sin A_3 = -\frac{\Delta}{R}.$$

John H. Lawlor.

ANSWERS TO QUERIES.

[67, p. 144, vol. v.] K is the centre, S, S' the foci of a conic inscribed in a triangle ABC, whose circumcentre is O, circumradius R, and nine-point centre N: prove that

$$OS\,.\,OS' = 2R\,.\,KN.$$

Several analytical proofs have been given, but the following variation may be of interest. Take R as unity, O as origin of Cartesian coordinates ξ, η; let $x=\xi+i\eta$, $y=\xi-i\eta$. Then the coordinates of the vertices A, B, C are of the form $x=a$, $y=1/a$ etc., where $|a|=1$.

* *N.B.*—The 2 is inserted merely for convenience in identifying, and is not an essential part of the standard equation.

The tangential equation to an inscribed conic is then

$$A(lb+m/b+n)(lc+m/c+n)+\ldots+\ldots=0. \qquad \ldots\ldots(1)$$

If (x_1, y_1) and (x_2, y_2) are the foci, the last equation (1) must be the same as

$$(lx_1+my_1+n)(lx_2+my_2+n)-klm=0. \qquad \ldots\ldots(2)$$

Hence $\qquad x_1x_2=Abc+Bca+Cab,$

$$y_1y_2=A/bc+B/ca+C/ab$$

and $\qquad x_1+x_2=A(b+c)+B(c+a)+C(a+b),$

$$1=A+B+C.$$

Thus $\qquad x_1+x_2=(a+b+c)(A+B+C)-(Aa+Bb+Cc)$

$$=a+b+c-abc(y_1y_2),$$

or $\qquad y_1y_2=\frac{1}{abc}[(a+b+c)-(x_1+x_2)]. \qquad \ldots\ldots(3)$

Now $\qquad x=\tfrac{1}{2}(a+b+c)$ gives the point N

and $\qquad x=\tfrac{1}{2}(x_1+x_2)$ gives the point K,

so that the absolute value of the right-hand of (3) is $2.KN$ or $2R.KN$: while that of the left is $OS.OS'$. T. J. I'A. B.

Anon. [A. 3. g.] *Note* 320. The equation $x^{2p+1}+ax^2-b=0$.

The series given are special cases of a generalization of Lagrange's theorem (see for instance my *Infinite Series*, p. 174, Ex. 30): the theorem states that if

$$y=a_2x^2+a_3x^3+\ldots,$$

then, provided that $|y|$ is small enough, there are two roots x_1, x_2 of this equation in x, which are also small. Further, that if $g(x)$ can be expanded in the form

$$c_0+c_1x+c_2x^2+\ldots,$$

then $\qquad g(x_1)+g(x_2)=2c_0+d_1y+d_2y^2+\ldots,$

where nd_n is the coefficient of $1/x$ in the expansion of $g'(x)/y^n$ in powers of x.

Taking $y=b/a$, $\alpha=1/a$, the equation proposed can be written

$$y=x^2+\alpha x^{2p+1},$$

and putting $g(x)=x$, the theorem leads to the series

$$x_1+x_2=-\alpha y^p-\frac{(3p+1)3p}{3\,!}\alpha^3y^{3p-1}-\frac{(5p+2)(5p+1)5p(5p-1)}{5\,!}\alpha^5y^{5p-2}-\ldots.$$

Similarly we find the series given for $x_1^2+x_2^2$, by taking $g(x)=x^2$.

As to the query about the interpretation when x_1, x_2 are not real; clearly x_1 and x_2 will be conjugate complex numbers, and so x_1+x_2, $x_1^2+x_2^2$ are still real. This will be the case if y is *negative* and not too large: the series will still converge, but if $x_1=\xi+i\eta$, $x_2=\xi-i\eta$ their sums will be 2ξ, $2(\xi^2-\eta^2)$, from which ξ, η can be found. Naturally, if y is larger the series may not converge even though the roots are real.

The condition of convergence seems to be $\alpha^2e^2|y|^{2p-1}<1$; or in the original notation, b^{2p-1}/a^{2p+1} must be less than $1/e^2$, in numerical value.

T. J. I'A. BROMWICH.

[68, p. 189.] *The circles whose diameters are the three diagonals of a quadrilateral are known to be coaxal. By what property of the four lines which form the quadrilateral is it possible to distinguish the cases when the common points of these circles are real or imaginary?*

Let ABC be the diagonal triangle; PP', QQ', RR' the extremities of the three diagonals. Let the circle on PP' as diameter cut the circumscribing circle of the triangle ABC in M and N. Since the two circles cut orthogonally it will be possible to draw a circle having A and B as inverse points to touch the circle on PP' as diameter in M, and another one to touch it

in N. Let these circles meet AB in FF' and GG' respectively; then it is clear that the circle on RR' as diameter will intersect the circle on PP' as diameter if it is not interior to one of the circles on FF' or GG' as diameter. This is the case if PR does not lie between PF and PF' or between PG and PG'. Now, it is easy to see that the line PF passes through the centre of one of the circles touching the sides of the triangle ABC, for since M lies on the circle ABC, we have

$$AM.BC \pm BM.CA \pm CM.AB=0;$$

also
$$\frac{AM}{BM}=\pm\frac{AF}{BF}, \quad \frac{CM}{BM}=\pm\frac{CP}{BP}.$$

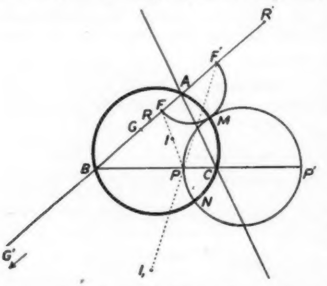
Now if $l\alpha+m\beta+n\gamma=0$ is the trilinear equation of PF, we have

$$\frac{l}{m}=\pm\frac{AF.BC}{BF.CA}, \quad \frac{n}{l}=\pm\frac{CP.AB}{BP.CA},$$

hence
$$l\pm m\pm n=0;$$

and so the line PF passes through the centre of one of the circles touching the sides of the triangle ABC.

The condition that PR should not lie between PF and PF' or between PG and PG' may now be expressed by saying that the product of the perpendiculars from the centres of the inscribed and escribed circles on PR must be positive.* This is easily seen to be the case because each of the four lines PF, PF', PG, PG' passes through the centre of one of these circles. The common points of the coaxal system of circles coincide when RR' coincide with either FF' or GG'. In this case each side of the quadrilateral passes through the centre of a circle touching the sides of the diagonal triangle.



A quadrilateral of this type may be obtained by reciprocating the corners of a triangle and the orthocentre with regard to a circle (or rectangular hyperbola) whose centre is on the nine points circle.

When the common points of the coaxal system of circles are real, any one of the four lines is the directrix of a conic circumscribing the diagonal triangle.

H. Bateman.

REVIEWS.

Mysticism in Modern Mathematics. By Hastings Berkeley. Pp. xii, 264. 8s. net. 1910. (Froude: Oxford University Press.)

Mr. Berkeley thinks that, as when the Pythagoreans were the foremost mathematicians, so even now the philosophy of mathematics has not wholly freed itself from mystical implications, and he believes (pp. iii.-iv.) that it is in general the case "that all ratiocinative processes, no matter what the subject, in which the current and continual substitution of symbols (of any kind) for concepts is a prime condition of the effective conduct of the process, are provocative of that attitude of mind" (cf. p. 5).

The book is divided into three parts: Part I. (pp. 3-49) deals with thought

* This means that an even number of the centres of these circles lie on one side of the line. It appears that if this is true for one side of the quadrilateral it must also be true for the others.

and its symbolic expression, Part II. (pp. 53-147) with imaginary quantities in algebra and imaginary loci in geometry, and Part III. (pp. 151-258) with metageometry. The first part contains an endeavour to arrive at a clear understanding of the mental attitude and processes involved in the use of language, and of symbolism in general, both as a means of intercommunication and as an instrument of reasoning (pp. 4, 250).

The discussions of the definition of number and of ordinal conceptions in arithmetic (pp. 53-56) are interesting, especially from a psychological point of view; but we miss a firm grasp of modern logical work on the principles of mathematics. Indeed, this appears to us to be the chief defect in, for the most part, an excellent book. The criticisms are, as a rule, sensible, unconventional, and in apparently unconscious agreement with the criticisms of modern mathematical logicians : thus, Mr. Berkeley points out and traces to its source (pp. 56, 99, 250) the mathematician's besetting sin of confusing sign with signification, and shows (pp. 85-87, 98) that, in developing algebra, we do not "generalize the concept of number" in the sense accepted by those mathematicians for whom the "principle of permanence" is one of the leading articles of their creed. And in this connexion it is instructive for teachers to read on p. 97 : "Anyone who can recall his school days, in particular his initiation into the mysteries of algebra, will, I doubt not, also recall the bewilderment produced in his mind by the authoritative divulgation of quantities less than no quantity and infinitely less than no quantity : a bewilderment which gradually yielded to the lethal effect of a sufficiently oft-repeated formula, accepted as significant with the trustfulness natural to youth and ignorance at the bidding of the pastor and master."

Mr. Berkeley characterises as "mystical" the belief, which seems to have been held by Cayley, among others, that expressions like "imaginary magnitude" and "imaginary locus" indicate modifications of the conceptions of magnitude and space (pp. 64-65, 130-131). The common-sense explanation of the radical axis of two circles, which passes through the points of intersection, or, if the circles do not intersect, is said to pass through the imaginary points of intersection is (p. 71; cf. pp. 129-130) : "The geometer perceives, in the construction common to the two opposed cases, a certain analogy; and this analogy is paradoxically, or by a violent metaphor, expressed in the statement that the line passes through the intersections, real or imaginary, of the circles. But then, so far as the expression 'imaginary points' alone is concerned, this is the philosophy of the matter. We require nothing more, save to recollect that we have expressed a real analogy by means of a verbal paradox, and that we must be careful, especially in the development of such an unusual mode of expression, not to lapse into the mystical by subsequently trying to read these expressions as if they were literal."

Mr. Berkeley's criticism (pp. 74-84) of the views of Boole and Dr. Whitehead on algebra are well worth study; but we have the impression that Mr. Berkeley has not grasped what seems to be the fact that Universal Algebra is not meant to be the expression of a view on what are denoted by algebraic symbols, but a method for the discovery of algebras.

With regard to non-Euclidean geometry, Mr. Berkeley's criticisms, although they seem just, do not appear to hit the mark. Modern mathematicians do not imagine that the purpose of geometry is to describe the space in which we live—it is quite likely that they would be mistaken if they thought this—but to affirm that, if a space has such-and-such properties, then it will possess such-and-such other properties. It is the *implication* that is asserted; the hypothesis and thesis are left doubtful.

In the main, we agree with Mr. Berkeley's criticisms. It appears to us quite certain that many mathematicians of eminence have fallen a prey to the mysticism which he attacks, usually with such vigour and correctness, and his book is a valuable contribution to the explaining of the paradoxical remark (pp. 6-7) that one hears so often, that some people have logical but "unmathematical" minds.

But, although Mr. Berkeley's criticisms often, unlike the various orthodox expositions of the foundations of mathematics, agree with the careful work of modern authors like Frege, Peano, and Russell, on the principles of mathe-

matics, Mr. Berkeley does not seem to be well acquainted with this work; and it is this work which most emphatically deserves the title of "modern," but seems to us, owing to its almost continual use of a powerful and subtle symbolic logic, and logic alone, quite free from mysticism.

PHILIP E. B. JOURDAIN.

Die Lehre von den geometrischen Verwandtschaften. By R. STURM. 4 vols. 8vo. (Leipzig: Teubner.)

A book, such as the present, conceived in a spirit not altogether sympathetic to many of the mathematicians of the present day is bound to have its critics. Its limitations are, indeed, evident at the first glance. But even those mathematicians who have long ceased to occupy themselves with the kind of investigations of which this work treats ought to welcome its appearance. Taken in conjunction with the "Liniengeometrie" it constitutes the record of a life-work.

The writer is an expert in his own department of the old careful German school which took some subject, or even some small portion of a subject, and completely mastered it. An account of the labours of such a man is not likely in all its details to interest a large audience. But, where publishers can be found to take a generous view of their duties towards science, the learned world ought to be unanimous in its applause.

The mathematician who turns to the book with the object of finding the last word on geometrical transformations is from the very character of the work doomed to disappointment. It has evidently been no part of the author's intention to give even a resumé of such work as, for example, that of Castelnuovo on the possibility of expressing a Cremona transformation as the product of quadratic transformations, though this work was epoch-making, constituting, as it does, the only existing rigid proof of this fundamental fact.

Again the circumstance that the writer definitely excludes all reference to any system of axioms prevents him presenting the principle of duality in what seems to many of us the most natural form. That the space with which Sturm concerns himself is three-dimensional space, was to be expected by those acquainted with his treatise on Line Geometry. Neither as a direct subject of thought, nor in respect to the light it throws on three-dimensional problems, has modern n-dimensional geometry influence on his writings. Probably the average reader of Sturm's work who had no other source of inspiration would fail to realise that every entity, which needed more than three coordinates to characterise it, would gain in, so to speak, visuality (Uebersichtlichkeit), if thought of in connection with higher space. If the Italians have shown that the geometry of such space can be treated with the same ease, and the same synthetic methods as those with which we are familiar, there still remains a large portion of the subject of geometry proper which has resisted all attempts to synthetise it. In fact, it might be contended that such efforts can from the very nature of things be only partially successful. The fact, therefore, that Sturm assumes no knowledge of analysis, and only a slight knowledge of algebra, limits at once, to an extraordinary degree, the extent of the field with which he occupies himself. Not only, what is self-evident, must the work of writers such as Hurwitz on the theory of correspondences fail to find mention, but the account even of Plucker's equations, which is one of the subjects treated, is bound inevitably to confine itself to the simplest cases which can occur.

On the other hand even the mathematician who has given up his life to work of the same character as Sturm's will find much in these four large volumes which is, to a greater or less extent, novel to him. The English reader, more perhaps than one of any other nationality, has much to gain by a study of the treatise before us. The writer works with concepts, and not with symbols, and no one who has read sufficient of the work to master its spirit is likely to devote pages of analysis to the proof of a result which to an Italian is almost intuitive, and he will acquire an insight into the reason why Cremona solved without effort the problem of the classification of quartic ruled surfaces, though the same problem led in the case even of so distinguished an analyst as Cayley to repeated failure.

Turning to the examination of the work in detail, it is unfortunate that the four pages of the table of contents give such inadequate information as to the subject-matter of the treatise. It is almost impossible to find out what precisely is in the

work without reading large portions of it. There is at the end of each volume no index, even of proper names. It is scarcely too much to say that the value of a treatise of this size, containing as it does eighteen hundred pages, is to many mathematicians reduced in this way almost to vanishing point. The conscientious reader who sets to work to study the book in detail is similarly handicapped by the fact that the references to previous and subsequent portions of the work are almost always absolutely vague. A favourite expression is that such and such a subject will be discussed "later." These faults might have been easily remedied had the proof-sheets been submitted to a colleague, or pupil, before being passed for press.

Sturm lays much stress on the fact that he has always given his proofs at length, calling attention to the fact that writers who have not done so have not infrequently fallen into error. But surely it was not necessary to state so frequently both the theorem and its dual. The explanation is evidently to be found in the fact that the treatise consists, as Sturm himself tells us, of his lectures at Breslau, with various additions. The verbal statement of both forms of the theorem no doubt helped to render the *lecture* more easily followed. The detail with which the elements are treated in the first part of the first volume is evidently due to the same cause.

Something would have been gained if certain parts of the work had been embodied in examples, or put into small print. That this would not be foreign to Sturm's method of treatment is evident here and there, *e.g.* on p. 97 of Vol. I, where he suggests to the reader to discuss the "feldduale Aufgabe."

In his treatment of imaginaries Sturm follows perhaps too closely the ultra-real school. It is interesting and instructive to have stated in real language the properties expressed analytically in terms of imaginaries. But the consequent loss of "Uebersichtlichkeit," as well as the fact that the possibility of so translating analytical statements is limited to properties of the simplest entities, seems to the reviewer to render it undesirable to lay too much stress on such a treatment. As soon as the *raison d'être* of the imaginaries is made clear geometrically, the argument in subsequent proofs would gain much in brevity and conciseness if imaginary entities were explicitly used. The mention of imaginaries leads one naturally to remark on Sturm's treatment of the space at infinity. The circumstance that there is no consequential introduction of axioms leads to the curious fact that, even as late as p. 127 of the first volume, Sturm writes "mit dem *später zu entwickelnden* Begriff der unendlich fernen Ebene des Raumes." With regard to Sturm's use of algebra, there is naturally a certain arbitrariness in his choice of when to use it and when not. Occasionally a proposition may be proved more simply by algebra, and yet be treated synthetically by writers of the severe synthetic school to which Sturm belongs. This is, of course, explained by the fact that algebra is only admitted by them on sufferance, and that its introduction is still of recent date.

There is a passing occasional reference to so-called problems of the second, third and fourth degrees. The attractiveness of the book to most readers would have been increased had it been part of Sturm's plan to devote more space to this subject.

To turn now to some of the contents of the book, the so-called problem of plane projectivity, to which one of the last chapters of the first volume is devoted, has a special interest to English readers. How masterly Sturm's treatment is, may be gathered by comparing this chapter with Cayley's work in the *Proceedings of the London Mathematical Society*, Vol. IV. p. 396, where Cayley's whole paper is little more than an analytical account of one of Sturm's results. The subject is an interesting one for other reasons. It has a bearing on the modern subject of measurements by means of photography (Photogrammetrie), a subject in which systematic research is much needed, and which has important applications, for instance, in Architecture, Geodesy, Crystallography, Meteorology and Military Science. The problem treated by Sturm is that in which we have before us two plane fields, supposed to be projective, and in which a finite number of pairs of images, $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$, are known. It is required to find a a pair of points A and B, such that the n rays AA_i are projective with the n rays BB_i. When n is 7, there are three possible positions for the point A, each determining uniquely the corresponding point B. Thus if one of these points is known, the other can be determined by means of the ruler alone. The construction of

both points, A and B, by means of the ruler and compasses alone is not in general possible, but there are some interesting special cases in which this can be done.

This problem occurs in practice when, for instance, we have two photographs of the same object from two different points of view.* The point A is then taken to be the image in the plane of the first photograph of the second point of view, while B is the image in the second plane of the first point of view. When A and B are known, and three pairs of corresponding points in the photographs, the ray in one picture corresponding to any given ray through the point A, or B, as the case may be, in the other picture can be constructed. This enables us to identify point for point on corresponding lines or curves. This is sufficient for many purposes, since corresponding lines and curves in the two photographs are usually recognisable by the eye. We are then able, if it be desired, to construct a model of the actual object to scale.

It is hardly necessary to mention that Cayley's name occurs several times, notably, of course, in connection with the Cayley-Brill formulae relating to correspondences on a non-unicursal curve. The English reader, again, will not fail to welcome the recognition given in this treatise to the work of Hirst, although the want already mentioned of an adequate index, or table of contents, would render it difficult for the student to find out the places where Sturm has followed our compatriot. The name Hirst occurs once in the index and once in the table of contents, viz.: in the mention of Hirst's Complex (I. 210). Besides this the reviewer notes that not only in dealing with the tetrahedral complex (I. 401), axial and central correlations (II. 221), and with the quadratic correspondence known as Hirst's, but more especially, as Sturm himself tells us, in the determination of general numerical formulae by means of special or degenerate cases, Hirst's influence is apparent. The reader who is interested in this last subject may be referred to Schubert's extremely readable treatise on "Abzählende Geometrie," in which also full justice is done to Hirst's share in founding this branch of mathematics. The aim of this Calculating Geometry is to answer all such questions as the following:—"How many geometrical elements of some particular form fulfil certain conditions," or "How many degrees of freedom has such an element?" The principal used so fruitfully by Hirst and Schubert, and reduced by the latter to a calculus, is that the number required may be determined by considering carefully chosen special cases. Thus, to take a trivial but typical example, we can assert that the number of points common to two plane curves of degrees m and n is mn, because this is the case when the curves degenerate into m and n straight lines respectively. Opinions as to the value, or rather as to the rigidity of results so obtained are divided, but most mathematicians are agreed that, as a means of investigation, the method is suggestive and fruitful.

Another Englishman whose work receives attention is H. J. S. Smith, who was the first to call attention to the interesting focal properties of a collineation.

Mathematicians who have at one time or other in their lives occupied themselves with elliptic functions, may, if so disposed, find occupation in expressing in analytical form the results given in the chapter on "Korrespondenzen auf Trägern vom Geschlechte 1." The analyst pure and simple can scarcely fail to be struck with the relative ease with which results stated in this chapter are obtained.

Perhaps the most disappointing chapters to one conversant with Italian literature are those on Cremona Transformations in the Plane and in Space. The subject itself is so fascinating when the aid of a little algebra is invoked, and becomes still more so when concepts are introduced closely connected with analysis, such as the entities which arise when two or more points approach to coincidence according to definite laws. Moreover, there are so many problems connected with these transformations that still require solving, that one turns to these chapters in the hope of getting some definite help towards their solution. It may be doubted, however, if anyone who has not read the greater part of the previous three volumes is likely to make much of these two chapters, and there is not sufficient indication in the chapters themselves as to the results given which are likely to be fruitful.

It is possible also that the title of the second part of the third volume will lead to some disappointment. It is called "Linear Systems of Curves and Surfaces," but it has no connection with the remarkable work of Brill and Noether, and of

* Finsterwalder, Jahresber, d- d. Mathvg. Bd. 6, Heft 2, pp. 7-11.

the Italian school, of which the latest representatives, whose work has become classical, are Castelnuovo and Enriques. The subject discussed by Sturm is of a different character, as would be anticipated by a reader of the preceding portions of the work. The results arrived at possess none the less a certain interest to mathematicians acquainted with that theory. For example, the number of points in which corresponding surfaces, one from each of six collinear nets of surfaces of orders $n_1, n_2, \ldots, n_6$, intersect is determined.

It is, perhaps, one of the defects of the work as a whole that the reader is likely to have difficulty in discovering from it in which direction he may profitably work. Sturm seems himself to have felt this, for he indicates in the preface to the fourth volume two such directions:—(1) construction of a theory of problems called by Hirst "Porismatic," which possess no solution, though the number of conditions is the right number; (2) the geometry on the general surface of the fourth degree. He suggests also that metrical properties generally have of late been neglected in comparison with projective, though this has certainly not been the case in England. He shares, too, with Klein the regret that in the last century the non-algebraic curves and surfaces fell into the background, though, as he suggests, in practice they are, perhaps, more important.

Many interesting properties of algebraic curves and surfaces of the lower orders, up to even the fifth, find their place in this treatise, and there is sufficient information on the subject of Hirst's porismatic problems to enable the conscientious student to undertake further research. On the whole we may certainly rejoice that Sturm has found time in his declining years to see this account of his lectures through the press, and we may feel sure that, if he had not succeeded in doing so, no book exactly like it would ever have seen the light. W. H. YOUNG.

Public School Arithmetic. By W. M. BAKER, M.A., and A. A. BOURNE, M.A. Pp. xii, 386. 3s. 6d.; or with Answers 4s. 6d. (Bell & Sons, 1910.)

The book before us is the usual English "Arithmetic," and as such can be well recommended. It contains the ordinary collection of rules, which, according to the preface, illustrate "those methods that have proved by experience to be the most successful under modern conditions." Commercial arithmetic, therefore, is in particular discussed, and the familiar arithmetical syllabus is well covered. To the teacher and the private student the methods and explanations will be very helpful, for they are, for the most part, clear, concise, and to the point. Whether the individual boy requires such a full text-book is, however, an open question; the general tendency nowadays perhaps being towards much oral teaching and a book of examples. Of the latter the authors have indeed made a splendid choice, and here, as they affirm, the chief merit of the book lies. "They (the examples) have been carefully graded; and in the selection of them it has been borne in mind that at certain stages a pupil learns more, in the way of method, from a number of questions which come out easily, than from some of those long laborious examples which have their use in other directions."

As regards the rest, it may be mentioned that fractions are attacked before decimals, and that the idea of ratio, upon which of course nearly the whole of the latter part of the book depends, is not given quite all the notice it deserves. The introduction to logarithms at the end could have done very well without the antilogarithm table to increase the general confusion brought about by the too early use of the word "logarithm"; and why, in these days of rural revival, should that unit of garden measurement—the rod—be relegated to a footnote at the bottom of the linear measure table? We have detected no misprints, and the general get-up of the book is of course excellent. H. G. MAYO.

Examination Papers in Arithmetic. By CHARLES PENDLEBURY, M.A. Pp. 212. 2s. 6d. 1910. (Bell & Sons, Ltd.)

This, though nominally a revised edition of Mr. Pendlebury's older book, is to all intents and purposes a new collection of examples. As might be expected, they have been admirably chosen, and will be of the greatest use in classes where the arithmetic lesson consists in the working of exercises supplemented when necessary by oral work from the teacher. Included are a hundred examination papers of eight assorted questions each, four hundred and one problems of the newest types, about two hundred and forty exercises in mensuration and loga-

rithms (unfortunately, perhaps, preceded by a list of formulae), and finally a whole host of recent public examination papers of a miscellaneous character. The general appearance and printing of the book are perfect. H. G. MAYO.

Elementary Projective Geometry. By A. G. PICKFORD. Pp. xii, 256. 4s. 1909. (Cambridge University Press.)

This attractive little introduction to Projective Geometry has distinct merits of its own, and, we are inclined to think, will soon give to the author pleasant proofs that it has justified its existence. The first four chapters deal respectively with cross ratio, involution, projective rows and pencils, and the circle. Projection in space is practically ignored. In "an elementary treatment of the subject I have avoided dependence on the use of points at infinity and imaginary points and lines; these will find place in a more advanced treatise." From this hint in the preface we presume that this volume is the first of a complete treatise which will cover wider ground than is offered for the consideration of the ordinary student. The fifth chapter defines the conic as the locus of the intersections of corresponding rays of two projective pencils not in perspective, and it is shown that the conic is the envelope of joins of corresponding points of two projective rows not in perspective, this property being also deduced from the focus and directrix definition of the curve. The converse is proved in Chap. viii. The chapters on polars and constructions of conics will strike many as showing the author's expository power at its best. Reciprocation and homology form the subject matter of the final chapters. In the chapter on the three conics the parabola is defined as the envelope of the join of corresponding points of two similar rows not in perspective. The envelope of the join or projective rows on TA, BT, whose vanishing points are I, J, is an ellipse or an hyperbola, as A lies or does not lie between T and I. But many of the proofs of the properties of conics in this chapter are those of the ordinary work on geometrical conics—a course which some teachers will no doubt prefer. The book is not overloaded with examples, but there are plenty for all practical purposes. The figures are clear. The full table of contents does not entirely supply the place of an index. There is a short list of errata, to which we may add that on p. 8, line 10 up, "when" or "if" seems to be omitted after "that." The form $\{AB, KL\}$ is used on p. 14, apparently without previous explanation.

An Elementary Treatise on Conic Sections by the Methods of Co-ordinate Geometry. By C. SMITH. New edition, revised and enlarged. Pp. 449. 7s. 6d. 1910. (Macmillan.) KEY to the same. 10s. 6d.

The merits of "Smith's Conics" are so well known to teachers of mathematics that there is no need for a reviewer to do more than indicate the main points in which improvements and additions have been made. The additions in all amount to nearly 100 pages, of which about a dozen form a new chapter on invariants. Amplifications of the text and additional examples bring the chapters on co-ordinates and the straight line from 42 pages to 54. The sections on cross ratio have been re-arranged and simplified, and there are two new sections added to those on involution at the expense of the old half dozen or so examples which are cut out. In the chapter on the circle practically the only changes are the addition of examples worked out in full, sectional examples, and miscellaneous examples for solution. The use of italics and leaded type throughout the book makes for rapidity of reference. There are a few new remarks in the sections dealing with the parabola, co-normal points are not forgotten, and the use of co-ordinates expressed in terms of a single variable is indicated. This chapter concludes with a couple of pages on envelopes, and 25 examples thereupon, while the miscellaneous examples are increased to 75. Sections on concyclic points and co-normal points on the ellipse are new, and here again the miscellaneous examples are now 75 in number. New worked-out examples seem to be the main changes in the chapter on the hyperbola; the miscellaneous examples are raised to 40, and then we have a second set of 30 miscellaneous examples. A few more examples are added to the chapter on the polar equation of a conic. A section on the equation of the tangents to a conic at the ends of a chord

precedes that on the equation of the director circle. Foci, directrices and eccentricity are found directly from the focus-directrix definition of the conic, the focoids are introduced and the old paragraph 193 (edition 1903) is re-written. Triangles inscribed in one conic and circumscribed about a coaxal is a useful addition to Chap. x. Paragraph 231 is an improvement on the old 229. The section on the circle of curvature is welcome to the private student. The old 234 is omitted, and 235 is unneccessary now that there is a chapter on invariants. There are a few alterations in Chap. xii., and in Chap. xiii. there is some rearrangement and many additions. The answers to examples, which contained hints very useful to the private student, have now dis-appeared. Determinants are now more freely employed in text and solutions, and the use of the solidus has saved space. From this imperfect summary even the teacher with slender purse will see that he cannot afford to ignore the claims of this new edition. The "Key" will be found invaluable to the private student, and many teachers will at times be glad to avail themselves of it.

A Manual of Geometry. By W. D. EGGAR. Part I. Pp. xiii, 160. Part II. Pp. xxiii, 156-323. 2s. 1910.

The list of requirements in Geometry for the Previous Examination at Cambridge has been accepted by Mr. Eggar as "the official guide to the study of geometrical truths," and has given him an opportunity of rearranging the subject-matter of his "Practical Exercises in Geometry," and of introducing the theorems along with the practical work. The great novelty for the pupil who takes Mr. Eggar's course is that he has to make his own text-book of formal proofs. The necessary hints are given, and after general discussion in class the boy writes out the proof in full for himself. This is corrected, a clean copy is made, and in this way the self-made text-book is gradually enriched for the purposes of the examination which is to come. There is a profusion of experimental illustration, and the usual apparatus of sets of questions, log. tables, etc., with an index which in completeness may serve as a model for works of the kind.

First Course in Calculus. By E. J. TOWNSEND and G. A. GOODENOUGH. Pp. x+466. 12s. 1908. (Bell & Sons.)

The authors are the Professor of Mathematics and the Associate Professor of Mechanical Engineering at the University of Illinois. Their book was first published in 1908, and now after two years, following a custom which seems to be growing prevalent, it reaches our hospitable shores. Though written primarily for the young engineer, the book will no doubt be found of use to ordinary students as "helpful and stimulating in showing the broad use of the calculus in practical problems." The method of limits has been used throughout. Chapters iii.-iv. deal with derivatives of algebraic functions and applications thereof. Chapter v. introduces anti-derivatives, integrals of x^n and u^n with illustrations from curves having given properties, rectilinear motion, and rotation about a fixed axis. Chapter vi. is on the differential notation, and the next, on differentiation of transcendental functions, concludes with integration by inspection and substitution, harmonic functions, and the compound interest law, with examples from physics, chemistry (*e.g.* the law of inversion of sugar), and electricity. Successive differentiation and integration, definite integrals, applications of integration to geometry and mechanics are the subject-matter of the next three chapters. They cover maxima and minima, concavity of curves, points of inflexion and curve tracing, with curvature, lengths and areas of curves, volumes, mean value, work of variable forces, and of expanding gases. Two chapters are devoted to special methods of integration and functions of two or more variables. Then follow multiple integrals with geometrical and physical applications—tangent planes, mean density, radius of gyration, centroids, stress, liquid pressure, and discharge through orifices. Thirty pages are given to infinite series, and 22 to envelopes, the osculating circle, involutes and evolutes, asymptotes, etc. A useful chapter deals with approximate integration, Simpson's rules, and the concluding 80 pages deal with ordinary differential equations and applications. Thus it will be seen that the order of development departs considerably from the normal, but there seems good reason for suspecting that the progress of

the student will be none the less rapid. The book is beautifully printed, the diagrams are excellent, and the selection of illustrative examples commendable. We have noticed a few misprints. In Ex. i. p. 193 the lower limit for the integral should be π; on p. 208 the coefficient of elasticity for wrought iron is given as 30,000,000, whereas on this side of the pond this is the coefficient for steel, that for wrought iron being more like 28,000,000; for 119, l. 2 p. 295 read 118; and on the same page in the first line of the solution to Ex. iv. for a^2b read πa^2b. The authors do not seem to care for the notation $x \rightarrow \infty$. On the whole they seem to prefer $x \doteq \infty$, but occasionally forget and revert to the old-fashioned $x = \infty$.

A School Algebra. By H. S. Hall. Part I. Pp. xi+299+xxxviii. 2s. 6d. 1910. (Macmillan.)

The present work is not, as Mr. Hall tells us in his preface, "a mere revision of any of the text-books on algebra with which my name is connected." The main changes are such as have been dictated by the experience of many years, and the arrangement of the pages is clearly the work of a practised constructor of class-books. The chapters on graphs has benefited by revision, and now, as the author claims, has both "plan and coherence." The sets of miscellaneous revision papers are "reasonably simple." Such alterations as we have noted are certainly for the better. If it be not too revolutionary a proposal, we should like to see the text of pp. 186-189 cut out and replaced by par. 218. All the 35 examples given on those pages can be solved with the utmost ease by repeated applications of: "If F divides A and B it also divides $mA \pm nB$." "Hall and Knight" is in the eyes of many teachers the best text-book for drilling the elements of algebra into beginners, and there is little doubt that the new volume will hold its own for a long time to come.

Practical Mathematics for Continuation Classes, embodying a Preparatory Course for Craftsmen. By T. Barr. Pp. viii+232. 2s. net. 1910. (Blackie.)

Mr. Barr's little book is designed to meet the requirements of artisans and others who, after leaving the continuation classes, will in all probability find a place in the technical colleges. A further advantage is recounted in the preface. "By the adoption of this manual the necessity of two or three text-books for those who study under the scheme of Glasgow, Govan and adjacent School boards . . . will be dispensed with." The book is divided into sections dealing in turn with Geometry, Algebra, Graphs, Mensuration, Logarithms and Trigonometry, and we are told that these subjects are thus kept in water-tight compartments "that the student may be able to judge what progress he is making." We notice, however, that there is some leakage from one compartment to another. For instance, a trigonometrical function is used without explanation on p. 130 in finding the equation to a straight line, and we should think the average student would have reached this stage of his work in graphs long before he has tackled the twelve pages of trigonometry which are to land him on the threshold of the technical college. The graph on p. 140 shows an indifferent approximation to the equivalent of 22 links in inches. Some canny Scot may question the accuracy of the statement on p. 79, viz., $+3 = +(+1)+(+1)+(+1)$, "that is, the positive unit is added to itself 3 times." He may think that if we add the unit to itself once we get 2, and if twice we get 3. The same doubt may assail his mind when he is told that 2 multiplied into itself 3 times is 8 (p. 77). Again, on p. 209 he will be puzzled to see "sine" and "cotangent" in leaded type—a dignity denied to the humble "cosine" and to the "tangent" on the next page. These are, however, but small matters. Continuation classes have come to stay, and if the students make as much progress in their mathematics as is indicated by the scope of this little book, the teachers in the technical colleges will have some reason to congratulate themselves. Taken as a whole, the book is carefully done, and we do not think that any very serious mischief will ensue if the "Glasgow, Govan and adjacent School Boards" do adopt it for their continuation classes.

The Student's Arithmetic. By W. M. Baker and Bourne. Pp. viii + 328 + 1. With answers. 2s. 6d. 1910. (Bell.)

This is a "shortened edition" of the *Public School Arithmetic* by the same authors, which, with answers, costs another couple of shillings. As the examples in both books are identical, those teachers who prefer their students to have the minimum of explanation will, by using it, gratify at once both the parents' desire for economy and their own pedagogic ideals. For the great fault of many text-books of the present day is that the path is made too easy. Even what we learn by our mistakes is often of far more value than what we acquire without the effort of thinking.

Practical Arithmetic. By R. S. Osborne. Pp. xi + 270. 2s. 6d. net. 1910. (Effingham Wilson.)

Mr. Osborne's experience as Lecturer on Commercial Methods and Practical Arithmetic at the City of London College is placed by the publication of this book at the disposal of teachers who have to get boys through the examinations of such bodies as The Chartered Institute of Secretaries, the Institute of Bankers, the Royal Society of Arts, etc. For admission to the arcana here revealed all that is required from the student is a "thorough knowledge of the very elementary rules of arithmetic." The first sections deal with decimals and approximations, ratio and proportion, the chain rule, the metric system and the like. After the use of the log. tables is mastered, the student proceeds to interest, discount, annuities, equation of payments, and our old friend, alligation. The 16th chapter will be found useful for reference. It deals with exchange calculations and most questions that arise from the consideration of the money market column of a financial paper—matters in which the teacher of to-day has, alas! too little of the knowledge that arises from immediate personal interest. The book is well printed, but, we would venture to suggest, suffers from a plethora of worked-out examples.

Second Stage Mathematics (with Modern Geometry). Edited by W. Briggs. Pp. 128 + 102 + 201 + 24. 3s. 6d. 1910. (University Tutorial Press.)

The geometry, algebra, and trigonometry requisite for candidates for the Second Stage of the "Science and Art Examinations" are faithfully dealt with in this volume by the unknown "specialists" of the Tutorial Press. Whoever they may be, it is clear that they are experienced teachers. If lengthy and detailed explanations, warning notices at the approach to every possible pitfall, and every use of typographical device to attract the attention and impress the memory are the marks of a good book, then this is one. The private student will be hard to please if he does not find between these covers all his requirements for the "Second Stage."

Conic Sections made Easy. By Saradakanta Gangopadhyaya. Pp. 97. 8 annas. 1909. (The Students' Library, Calcutta.)

This little book is intended in the first place for candidates for the Intermediate Examination of the Calcutta University. It consists of two chapters dealing respectively with the parabola and the ellipse. "I have preferred to treat the subject after the manner of Euclid, with whose mode of treatment the student is so very familiar, and have taken the bold step of replacing some of the existing proofs by new ones, which have seemed to me to be more easily intelligible." Hence some of the proofs are longer than usual. Elegance may be later attained, but at the outset of the student's acquaintance with the conic sections the author considers that he is justified in sacrificing brevity to simplicity. The English teacher of geometrical conics might with advantage consider the obvious advantage that is offered by this approach to a branch of mathematics which is not as popular as it might be. The explanations throughout show the hand of a teacher of experience. The exercises are carefully selected and graduated. The author's name is unfortunately mis-spelled on the cover.

The Student's Matriculation Geometry. By SARADAKANTA GANGOPADHYAYA. Books I-IV. Pp. xviii, 348. Re. 1.4. 1909. (The Students' Library, Calcutta.)

The sale of more than 2000 copies of this little book in twelve months would seem to show that, at any rate as far as "strict conformity with the Calcutta University Syllabus for the Matriculation Examination" is concerned, it has justified its existence. It is divided into four books treating in turn of: angles, lines, and rectilineal figures; areas and loci; the circle; practical geometry. While it is not on all fours with the introductions to geometry to which we have been accustomed in this country during the last few years, there are some novelties which are of interest to the teacher. In some respects British opinion is perhaps not ripe enough to follow all Mr. Gangopadhyaya's innovations. For instance, among the instructions to the students we find the following:

(5) Do not rest satisfied unless and until you fully understand the "how" and the "why" of every step of the proof, *so that you may not lose marks*. . . . (The italics are ours.)

(7) If you are desirous of being a credit to yourselves and a source of satisfaction to your parents and teachers, remember

WHAT A MAN HAS DONE A MAN MAY DO,
and
PROCRASTINATION IS THE THIEF OF TIME.

The table on p. 78 showing the nine converse theorems of I. 4 should, we think, be found interesting to boys. The direct proof of I. 25 by superposition, p. 105, was discovered by Mr. Justice Asutosh Mukhopadhyaya, the Vice-Chancellor of Calcutta University, at the age of eleven. A note on I. 47 gives reasons for the belief that the theorem of Pythagoras was known in very early times to the Hindus. Indeed, attention is drawn throughout to theorems which can be attributed to Hindu mathematicians. The proof of I. 19 given by the author is as follows: In the triangle ABC let $\hat{B}$ be greater than $\hat{C}$. Make $A\hat{B}D = \hat{C}$ cutting AC in D. Bisect $D\hat{B}C$ by BE meeting AC in E. Then $A\hat{B}E = A\hat{B}D + D\hat{B}E = A\hat{C}B + C\hat{B}E = A\hat{E}B$. Then $AE = AB$. But $AC > AE$, therefore $AC > AB$. Oral questions in plenty are given.

Descriptive Geometry. By G. ANTHONY and G. ASHLEY. Pp. 130; with 34 full page plates and 195 figures. 6s. 1909. (Heath.)

This well-got-up little volume should form an admirable introduction to the mysteries of the different systems of projection. Portions of it should, of course, form part of the work undertaken by every student before beginning solid geometry. A chapter of 12 pp. dealing with definitions and first principles is followed by one of 50 pp. on points, lines and planes. Each problem worked out gives briefly the geometrical principles that are to be applied, a sketch of the method suitable for the particular case, and a construction in full, followed by a statement of the checks advisable and any notes and comments that seem appropriate. Classification and Generation of Surfaces, including convolute warped surfaces and surfaces of double curvature, are next dealt with; tangent planes take ten pages; intersections of planes and surfaces lead to development and intersections of surfaces. The final chapter treats of warped surfaces, and the whole concludes with about fifty examples for solution. It seems an excellent book. We might, *en passant*, suggest that e^p is a somewhat awkward symbol to denote a point in a diagram.

Practical Electricity and Magnetism. By R. E. STEEL. Pp. viii, 171. 2s. 1910. (Bell.)

This little volume is a companion to Mr. Sinclair's Handbooks on Practical Physics. It consists of instructions for 114 experiments, for the most part in magnetism and current electricity, with whatever is necessary in the way of discussion. Useful sets of questions are appended to each chapter. Here and there one notes an experiment which seems to be rather beyond the average beginner, but we must bow to the wide experience of Mr. Steel.

New Plane and Solid Geometry. By W. Wells and R. L. Short. Pp. v+298. 1909. (Heath.)

The makers of manuals on geometry may get some useful hints from this volume. In 1847 William Pickering published for Mr. Oliver Byrne, Surveyor of Her Majesty's Settlements in the Falkland Islands, a book entitled *The First Six Books of Euclid, in which coloured diagrams and symbols are used instead of letters for the greater ease of learners.* Mr. Byrne's example did not prove to be contagious, but we suppose that there are few teachers who at some time or another have not used coloured chalks in their endeavour to press home geometrical truths.

Segnius irritant animos demissa per aurem
Quam quae sunt oculis subjecta fidelibus.

Messrs. Wells and Short give their readers eight coloured plates in addition to the usual figures. Lines of the same colour in a pair of triangles will be equal lines. The given and required lines are in all cases heavy, and lines of construction are dotted. All figures are not given. Under each figure the instructions for construction appear in small print. Nor do all the proofs appear in full—after a certain stage. Hints are provided, and the student fills up the proof himself. The steps of all proofs are numbered. The subject-matter of the first six of Euclid's books is compressed into 174 pages, the authors' five "books" being headed as follows: rectilinear figures; the circle; proportion, similar polygons; areas of polygons; regular polygons, measurement of the circle, loci. Then come four "books" on Solid Geometry: lines and planes in space, diedral angles, polyedral angles; polyedrons; cylinder and cone; sphere. It is evident that a considerable amount of thought has been given to the construction of this manual, and we think that the teacher will find it both instructive and suggestive on many points of doubt and difficulty.

A General Text-Book of Elementary Algebra. Exercises. Book I. to Quadratic Equations. By A. E. Layng. Pp. 88+xxxiv. 1s. 6d. 1910. (Blackie.)

This seems to be a reprint of the examples in *A General Text-book.* They are carefully graduated, and should prove useful to teachers who prefer their boys to begin the subject without the aid of a text-book.

A Class Book of Trigonometry. By C. Davison. Pp. viii, 200. 3s. 1910. (Cambridge University Press.)

It is very disheartening to find that the spirit animating the reforming party is so slow in penetrating into some quarters. Considering the stage we have now reached, we confess that it is difficult to see any reason for the existence of this "Class-book of Trigonometry." It may be considered by some to be a suitable introduction to the subject for the future professional mathematician. But to the majority of teachers who know what is really wanted for ordinary boys in the school of to-day, it will seem nothing less than a retrograde step to publish another volume of the ancient and discredited type. It is short and simple. So are many rivals. It leaves out what many other books leave out, or what every teacher can of his own accord dispense with. It is to be regretted that the solution of triangles is postponed so late as it is in this volume. Nor is it easy to characterise the somewhat misleading statement that it contains "easy problems in surveying," seeing that no practical work at all is set. The questions set on "surveying" are too few for the teacher to select from, and this in itself is sufficient to show how remote is the author's conception of the place of the subject in mathematical education from that of the Mathematical Association. Type, figures, setting-up, and all the rest of it have the distinction common to most books from the Cambridge University Press. But that does not seem to us to provide a *raison d'être* for a new Trigonometry at 3s., in which teachers will look in vain for any appreciation of the trend of informed opinion as to the proper presentation of this subject in the mathematical training of the average boy.

BOOKS, ETC., RECEIVED.

The American Journal of Mathematics. Edited by F. MORLEY. Oct. 1910. Vol. XXXII. No. 4. $1.50. (Johns Hopkins Press.)

q-Difference Equations. REV. F. H. JACKSON. *On the Relation between the Sum-Formulas of Hölder and Cesàro.* W. B. FORD. *Sur un Exemple de Fonction Analytique partout continue.* D. POMPEIU. *Symmetric Binary Forms and Involutions.* A. B. COBLE. *Systems of Tautochrones in a general Field of Force.* H. W. REDDICK. *The General Transformation Theory of Differential Elements.* E. KASNER.

School Science and Mathematics. Edited by C. H. SMITH. Nov. 1910. 25 c. (Smith & Turton, Chicago.)

The Use of Numbers in Measurement. E. B. SKINNER. *Report on Real Applied Problems in Algebra and Geometry.* J. F. MILLS. *Formulae for rational Right-angled Triangles.* *Practical Pedagogical Implications of Preliminary Report of National Committee on Geometry Syllabus.* W. BETZ.

The Mathematics Teacher. Vol. III. No. 1. Sept. 1910. (Published quarterly by the Association of Teachers of Mathematics for the Middle States and Maryland.)

On the Curriculum of Mathematics. I. J. SCHWATT. *Formal Discipline.* W. H. METZLER. *A shortened Form of Synthetic Division and some of its Applications.* EUG. R. SMITH. *A Simplification in Elementary Trigonometry.* W. H. JACKSON. *Solid Geometry.* H. F. HART. *Some remarks on Approximate Computation.* M. J. BABB. *A Generalised Definition of Limit.* E. D. ROE (Junr.).

Why do we study Mathematics? A Philosophical and Historical Retrospect. (Address delivered before the Secondary Mathematics Section of the National Education Association.) By. T. J. McCORMACK. (The Torch Press, Cedar Rapids, Iowa.) 1910.

Plane and Spherical Trigonometry. By D. A. ROTHROCK. Pp. xii, 148+xiv+100. 6s. net. 1910. (The Macmillan Co.)

Key to Hall and Stevens' School Arithmetic. Part II. Pp. 247. 6s. 1910. (Macmillan.)

Analytic Geometry. By N. C. RIGGS. Pp. xii+294. 6s. 6d. net. 1910. (The Macmillan Co.)

The Journal of the Association of Teachers in Technical Institutions. Edited by P. ABBOTT, M.A. April, 1910. 1s. (St. Bride's Press.)

The Course in Pure Mathematics. P. COLEMAN. *Mathematics applied to Engineering.* T. B. MORLEY. *A Method of introducing the Calculus.* R. H. DUNCAN.

An Elementary Treatise on Coordinate Geometry of Three Dimensions. By R. J. T. BELL. Pp. xvi+355. 10s. net. 1910. (Macmillan.)

A School Course of Heat. By R. H. SCARLETT. Pp. xvi+300. 3s. 6d. 1910 (Longmans.)

Annals of Mathematics. Edited by ORMOND STONE and others. Oct. 1910. Vol. XII. Ser. 2. No. 1. 2s. (Longmans, Green.)

The Straight Line Solutions of the Problem of the n Bodies. F. R. MOULTON. *On Semi-Analytic Functions of Two Variables.* M. BÔCHER. *Some Theorems concerning Systems of Linear Partial Differential Expressions.* W. J. BERRY. *Some Circles associated with Concyclic Points.* J. L. COOLIDGE. *On a Method for the Summation of Series.* R. E. GLEASON.

Calculus Made Easy. Being a very-simplest Introduction to those Beautiful Methods of Reckoning which are generally called by the Terrifying Names of the Differential and the Integral Calculus. By F.R.S. Pp. 178. 2s. net. 1910. (Macmillan.)

Proceedings of the Tokyo Mathematico-Physical Society. July, 1910.

Zur Theorie der electromagnetischen Vorgänge in bewegten Korpern. J. ISHIWARA. *Bemerkung über der Fortpflanzung des Lichtes in bewegten Medien.* J. ISHIWARA. *Zur Dynamik bewegter Systeme.* J. ISHIWARA. *Motion of a Projectile in a Resisting Medium.* S. YOKOTA.

Annaes Scientificos da Academia Polytechnica do Porto. Edited by GOMES TEIXEIRA. Vol. V. No. 3.

Essai d'une théorie analytique des lignes non-euclidiennes. GEMINANI PIRONDINI.

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